Lab quiz 9

1. Is the series $\sum_{n=1}^{\infty} \frac{2n^3-6}{5n^7+3n^4-2}$

   (A) Convergent

   (B) Divergent

   Solution:

   (Can also be done by BCT but you need to careful about showing the inequality correctly. So I prefer LCT in these cases.)

$$\text{Let } \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n^3-6}{5n^7+3n^4-2} \text{ and } \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^7-6n^4}{5n^7+3n^4-2} = \frac{2}{5} > 0$$

   Now $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^4}$ is convergent by p-series test as p=4, so by limit comparison test the series $\sum_{n=1}^{\infty} a_n$ is convergent.

2. Is the series $\sum_{n=1}^{\infty} \left(\sqrt{n} - \sqrt{n-1}\right)^n$

(A) Convergent

(B) Divergent

Solution:

$$\begin{aligned}
\lim_{n\to\infty} \left(\left(\sqrt{n} - \sqrt{n-1}\right)^n\right)^{\frac{1}{n}} &= \lim_{n\to\infty} \left(\sqrt{n} - \sqrt{n-1}\right) \\
&= \lim_{n\to\infty} \frac{(\sqrt{n} - \sqrt{n-1})(\sqrt{n} + \sqrt{n-1})}{\sqrt{n} + \sqrt{n-1}} \\
&= \lim_{n\to\infty} \frac{(\sqrt{n})^2 - (\sqrt{n-1})^2}{\sqrt{n} + \sqrt{n-1}} \\
&= \lim_{n\to\infty} \frac{(n) - (n-1)}{\sqrt{n} + \sqrt{n-1}} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{n} + \sqrt{n-1}} \\
&= 0 < 1
\end{aligned}$$

Hence the series is convergent by root test.

3. Is the series $\sum_{n=1}^{\infty} \frac{2n+\sqrt{n}}{n^3+3\sqrt{n}}$

(A) Convergent

(B) Divergent

Solution:

Method 1:

Convince yourself about each step of the inequality. We can increase a fraction if we increase the numerator or decrease the denominator.

$$\sum_{n=1}^{\infty} \frac{2n+\sqrt{n}}{n^3+3\sqrt{n}} \leq \sum_{n=1}^{\infty} \frac{2n+\sqrt{n}}{n^3} \leq \sum_{n=1}^{\infty} \frac{2n+n}{n^3} = \sum_{n=1}^{\infty} \frac{3n}{n^3} = \sum_{n=1}^{\infty} \frac{3}{n^2}$$

Now $\sum_{n=1}^{\infty} \frac{3}{n^2}$ is convergent by p-series test as p=2.
Hence the series is convergent by basic comparison test.

Method 2:

$$\text{Let } \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n+\sqrt{n}}{n^3+3\sqrt{n}} \text{ and } \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^3+n^{3/2}}{n^3+3\sqrt{n}} = 2 > 0$$

Now $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by p-series test as p=2, so by limit comparison test the series $\sum_{n=1}^{\infty} a_n$ is convergent.

4. Is the series $\sum_{n=1}^{\infty} \frac{\left(1+\frac{2}{n}\right)^{n^2}}{e^n}$

(A) Convergent

(B) Divergent

Solution:

$$\lim_{n\to\infty} \left(\frac{\left(1+\frac{2}{n}\right)^{n^2}}{e^n} \right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)^{n}}{e} = \frac{e^2}{e} = e > 1$$

Hence the series is [redacted] by root test.

divergent

5. Which of the following series is convergent?

(A) $\sum_{n=1}^{\infty} [2 + (-1)^n]$
Divergent (Easy to check)

(B) $\sum_{n=1}^{\infty} \frac{n^n}{n+1}$
Method 1:
If $a_n = \frac{n^n}{n+1}$ then

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n+2} . \frac{n+1}{n^n} \\
&= \lim_{n\to\infty} \frac{(n+1)^n(n+1)}{n^n(n+2)} \\
&= \lim_{n\to\infty} \left(\left(\frac{n+1}{n} \right)^n . \frac{n+1}{n+2} \right) \\
&= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n . \lim_{n\to\infty} \frac{n+1}{n+2} \\
&= e.1 = e > 1.
\end{aligned}$$

Hence divergent by ratio test.(Can also be done by root test)

Method 2:
$\lim_{n\to\infty} \frac{n^n}{n+1} = \text{DNE} \not\to 0$
As the limit does not go to 0 so divergent by basic divergence test.

(C) $\sum_{n=1}^{\infty} \frac{2^n}{(n+1)!}$
If $a_n = \frac{2^n}{(n+1)!}$ then

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{(2)^{n+1}}{(n+2)} . \frac{(n+1)!}{2^n} \\
&= \lim_{n\to\infty} \frac{(2)^n(2)(n+1)!}{2^n(n+2)!} \\
&= \lim_{n\to\infty} 2\frac{1.2.3......n.(n+1)}{1.2.3......n.(n+1).(n+2)} \\
&= \lim_{n\to\infty} \frac{2}{n+2} \\
&= 0 < 1
\end{aligned}$$

Hence convergent by ratio test

(D) $\sum\limits_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$

Method 1:

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1} \geq \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+n} = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n} = \sum_{n=1}^{\infty} \frac{1}{2n^{1/2}}$$

As $\sum\limits_{n=1}^{\infty} \frac{1}{2n^{1/2}}$ is divergent by p-series test as $p=\frac{1}{2} < 1$, hence the series is divergent by basic comparison test.

Method 2:

$$\text{Let } \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1} \text{ and } \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1 > 0$$

Now $\sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent by p-series test as $p=\frac{1}{2}$, so by limit comparison test the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

(E) $\sum\limits_{n=1}^{\infty} \frac{2^{3n+1}}{7^n}$

$$\sum_{n=1}^{\infty} \frac{2^{3n+1}}{7^n} = \sum_{n=1}^{\infty} \frac{2^{3n}.2}{7^n} = \sum_{n=1}^{\infty} \left(\frac{8}{7}\right)^n .2$$

Now this is a geometric series with $r=\frac{8}{7}$ so $|r|=\frac{8}{7} > 1$, thus divergent by geometric series test.

(F) No option is correct or more than one options are correct