Math 3321

Homogeneous Systems of Linear Differential Equations. Part III

University of Houston

Lecture 25

Outline

1. Homogeneous Systems - Repeated Eigenvalues

2. Non-Homogeneous Systems (sketch)

Homogeneous Systems with Constant Coefficients

Recall:

Corollary

Consider the homogeneous system of n equations with constant coefficients

$$x' = Ax$$

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If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are n **distinct eigenvalues** of A with corresponding eigenvectors $v_1, v_2, \cdots, v_n$, then

$$x_1 = e^{\lambda_1 t} v_1,\ x_2 = e^{\lambda_2 t} v_2,\ \cdots, x_n = e^{\lambda_n t} v_n$$

is a fundamental set of solutions of the system and

$$x(t) = C_1 x_1 + C_2 x_2 + \cdots + C_n x_n$$

is the general solution.

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This holds for both real and complex distinct eigenvalues.

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In the last lectures, I illustrated the case where the $n \times n$ matrix of coefficient A has n **real or complex distinct eigenvalues**.

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In this lecture, I consider the case where there are **no** n **distinct eigenvalues.**

Example 1: Find a fundamental set of solutions of

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We compute the eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix}$$

$$= 16 + 12\lambda - \lambda^3 = -(\lambda - 4)(\lambda + 2)^2.$$

We find the eigenvalues: $\lambda_1 = 4,\ \lambda_2 = \lambda_3 = -2$

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We find the eigenvalues: $\lambda_1 = 4,\ \lambda_2 = \lambda_3 = -2$

Note: the eigenvalue $\lambda_2 = \lambda_3 = -2$ is repeated!

Homogeneous Systems with Constant Coefficients

For $\lambda_1 = 4$, we examine $(A - 4I) = \begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix}$

Solve:

$$(A - 4I)x = \begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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The augmented matrix row reduces to

$$\left(\begin{array}{ccc|c} -3 & -3 & 3 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -12 & 6 & 0 \\ 0 & -12 & 6 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

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we obtain the eigenvector $v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

Homogeneous Systems with Constant Coefficients

We next examine the eigenvalues $\lambda_2 = \lambda_3 = -2$:

To find the eigenvector(s), we examine $A - (-2)I = \begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix}$

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The augmented matrix row reduces to

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Note that this matrix has rank 1.

Homogeneous Systems with Constant Coefficients

Solution set:

$$x_1 = x_2 - x_3,\ x_2,\ x_3 \quad \text{arbitrary}$$

Set $x_2 = 1,\ x_3 = 0:\quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix};$

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Fundamental set:

$$\left\{ e^{4t}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},\ e^{-2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ e^{-2t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

In this example, **even with repeated eigenvalues, we have a full set of eigenvectors**.

Example 2: Find a fundamental set of solutions of
$x' = \begin{pmatrix} -4 & 1 \\ -4 & 0 \end{pmatrix} x.$

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Eigenvalues:

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Homogeneous Systems - Repeated Eigenvalues

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$$(A - (-2)I)x = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -2 & 1 & 0 \\ -4 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

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Problem: Only one eigenvector and only one solution!

Unlike Example 1, **we do not have a full set of eigenvectors. We need another solution to be able to write the general solution of the system.**

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We compute the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 6 & 2 \\ 0 & -1-\lambda & -8 \\ 1 & 0 & -2-\lambda \end{vmatrix}$$

$$= -36 + 15\lambda + 2\lambda^2 - \lambda^3 = -(\lambda+4)(\lambda-3)^2.$$

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We find the eigenvalues: $\lambda_1 = -4,\ \lambda_2 = \lambda_3 = 3.$

We have repeated eigenvalues $\lambda_2 = \lambda_3 = 3$

Homogeneous Systems - Repeated Eigenvalues

We compute the eigenvector for the eigenvalue $\lambda_1 = -4$.
Need to solve the homogeneous system:

$$(A + 4I)x = \begin{pmatrix} 9 & 6 & 2 \\ 0 & 3 & -8 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 9 & 6 & 2 & 0 \\ 0 & 3 & -8 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 3 & -8 & 0 \\ 0 & 6 & -18 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Hence we have the solution: $x_1 = -2x_3,\ x_2 = \frac{8}{3}\,x_3, \quad x_3$ arbitrary

Setting $x_3 = -3$, we obtian the eigenvector $v_1 = \begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}$

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Hence we obtain the solution: $x_1 = e^{-4t} \begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}$

Homogeneous Systems - Repeated Eigenvalues

We next compute the eigenvector(s) for the eigenvalue with multiplicity 2, $\lambda_2 = \lambda_3 = 3$:
Need to solve the homogeneous system:

$$(A - 3I)x = \begin{pmatrix} 2 & 6 & 2 \\ 0 & -4 & -8 \\ 1 & 0 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & 6 & 2 & 0 \\ 0 & -4 & -8 & 0 \\ 1 & 0 & -5 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & -4 & -8 & 0 \\ 2 & 6 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 6 & 2 & 0 \\ 0 & -4 & -8 & 0 \\ 1 & 0 & -5 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Homogeneous Systems - Repeated Eigenvalues

Hence we have the solution:

$x_1 = 5x_3,\; x_2 = -2x_3,\; x_3$ arbitrary

Set $x_3 = 1$: $v_2 = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$

Homogeneous Systems - Repeated Eigenvalues

Hence we have the solution:

$x_1 = 5x_3, \; x_2 = -2x_3, \; x_3$ arbitrary

Set $x_3 = 1$: $v_2 = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$

NOTE: Only one eigenvector!

Solutions:

$$x_1 = e^{-4t} \begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}, \quad x_2 = e^{3t} \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}.$$

Hence we have the solution:
$x_1 = 5x_3,\ x_2 = -2x_3,\ x_3$ arbitrary

Set $x_3 = 1$: $v_2 = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$

NOTE: Only one eigenvector!

Solutions:

$$x_1 = e^{-4t}\begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}, \quad x_2 = e^{3t}\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}.$$

Problem: Only two solutions!
We need a third solution x_3 which is independent of $x_1,\ x_2$. to be able to write the general solution of the system.

Homogeneous Systems - Repeated Eigenvalues

To find the extra solution, we are going to take a **lesson from linear differential equations**.

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Characteristic equation:

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Fundamental set: $\left\{e^{3t},\ e^{-2t},\ te^{-2t}\right\}$

Homogeneous Systems - Repeated Eigenvalues

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We find the eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 12 & 8 & -1-\lambda \end{vmatrix} = -\lambda^3 - \lambda^2 + 8\lambda + 12\lambda$$

giving the characteristic equation:

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Hence we have the eigenvalues: $\lambda_1 = 3, \quad \lambda_2 = \lambda_3 = -2$

Homogeneous Systems - Repeated Eigenvalues

we have the Fundamental set:

$$x_1 = e^{3t}\begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}, \quad x_2 = e^{-2t}\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix},$$

$$x_3 = e^{-2t}\begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + te^{-2t}\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

Homogeneous Systems - Repeated Eigenvalues

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Note that x_1 and x_2 are the solution obtained from the eigenvalues and eigenvectors of λ_1 and λ_2. **What about the solution vector x_3?**

Note that $\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$ in x_3 is an eigenvector for -2

What is the vector $\begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}$**?**

Homogeneous Systems - Repeated Eigenvalues

$$(A-(-2)I)\begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 12 & 8 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

Hence:

$A-(-2I)$ maps $w = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}$ onto the eigenvector $v = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$.

Homogeneous Systems - Repeated Eigenvalues

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The vector $w = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}$ is called a **generalized eigenvector**.

Homogeneous Systems - Repeated Eigenvalues

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The third solution has the form

$$x_3 = e^{-2t}w + te^{-2t}v$$

Solution of Homogeneous Systems with 2 repeated eigenvalues

Given $x' = Ax$, suppose that A has an eigenvalue λ of multiplicity two. Then exactly one of the following holds:

Homogeneous Systems - Repeated Eigenvalues

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1. λ has two linearly independent eigenvectors, v_1 and v_2 and the corresponding linearly independent solution vectors of the differential system are

$$x_1(t) = e^{\lambda t} v_1 \quad \text{and} \quad x_2(t) = e^{\lambda t} v_2.$$

Homogeneous Systems - Repeated Eigenvalues

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$$x_1(t) = e^{\lambda t} v_1 \quad \text{and} \quad x_2(t) = e^{\lambda t} v_2.$$

2. λ has only one eigenvector v. Then a linearly independent pair of solution vectors corresponding to λ is:

$$x_1(t) = e^{\lambda t} v, \quad x_2(t) = e^{\lambda t} w + t e^{\lambda t} v$$

where w is a vector that satisfies $(A - \lambda I)w = v$.
The vector w is called a **generalized eigenvector** corresponding to the eigenvalue λ.

The first situation of the general method presented in the slide above was illustrated by **Example 1.**

The second situation of the general method presented in the slide above was illustrated by **Example 2** and **Example 3.**

We are now going to revisit **Example 2** and **Example 3** to compute the missing solution of the homogeneous linear system.

Back to Example 2.

$$x' = \begin{pmatrix} -4 & 1 \\ -4 & 0 \end{pmatrix} x$$

We found that corresponding to the eigenvalue $\lambda_1 = \lambda_2 = 2$, there is 1 eigenvector $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and a solution:

$$x_1 = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The second solution has the form

$$x_2 = e^{2t} w + t e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Homogeneous Systems - Repeated Eigenvalues

To find w, we solve:

$$(A-(-2)I)w = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -2 & 1 & 1 \\ -4 & 2 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1/2 & -1/2 \\ 0 & 0 & 0 \end{array}\right)$$

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For $w_2 = 3$, we get $w_1 = 1$, hence $w = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

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Hence: $w_1 = 1/2 w_2 - 1/2$, w_2 arbitrary.

For $w_2 = 3$, we get $w_1 = 1$, hence $w = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Hence we obtain the Fundamental Set:

$$x_1 = e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad x_2 = e^{-2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + te^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Back to Example 3.

$$x' = \begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{pmatrix} x.$$

We need to find the second solutions corresponding to the repeated eigenvalue $\lambda_2 = \lambda_3 = 3$
Such solution has the form

$$x_3 = e^{3t} w + t e^{3t} v$$

To find w, solve

$$(A - 3I)w = \begin{pmatrix} 2 & 6 & 2 \\ 0 & -4 & -8 \\ 1 & 0 & -5 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

Homogeneous Systems - Repeated Eigenvalues

$$\left(\begin{array}{rrr|r} 2 & 6 & 2 & 5 \\ 0 & -4 & -8 & -2 \\ 1 & 0 & -5 & 1 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & -5 & 1 \\ 0 & -4 & -8 & -2 \\ 2 & 6 & 2 & 5 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & -5 & 1 \\ 0 & 1 & 2 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Solution set:

$$w_1 = 1 + 5w_3,\ w_2 = \frac{1}{2} - 2w_3,\ w_3 \text{ arbitrary}$$

Set $w_3 = 0$: $\quad w = \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix}$

Homogeneous Systems - Repeated Eigenvalues

$$\left(\begin{array}{rrr|r} 2 & 6 & 2 & 5 \\ 0 & -4 & -8 & -2 \\ 1 & 0 & -5 & 1 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & -5 & 1 \\ 0 & -4 & -8 & -2 \\ 2 & 6 & 2 & 5 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & -5 & 1 \\ 0 & 1 & 2 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Solution set:

$$w_1 = 1 + 5w_3,\ w_2 = \frac{1}{2} - 2w_3,\ w_3 \text{ arbitrary}$$

Set $w_3 = 0$: $\quad w = \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix}$

The third solution is:

$$x_3 = e^{3t}\begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix} + te^{3t}\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

Homogeneous Systems - Repeated Eigenvalues

$$\left(\begin{array}{rrr|r} 2 & 6 & 2 & 5 \\ 0 & -4 & -8 & -2 \\ 1 & 0 & -5 & 1 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 1 & 0 & -5 & 1 \\ 0 & -4 & -8 & -2 \\ 2 & 6 & 2 & 5 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 1 & 0 & -5 & 1 \\ 0 & 1 & 2 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Solution set:

$$w_1 = 1 + 5w_3,\ w_2 = \frac{1}{2} - 2w_3,\ w_3 \text{ arbitrary}$$

Set $w_3 = 0$: $\quad w = \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix}$

The third solution is:

$$x_3 = e^{3t}\begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix} + te^{3t}\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

Fundamental set:

$$x_1 = e^{-4t}\begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix},\ x_2 = e^{3t}\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix},\ x_3 = e^{3t}\begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix} + te^{3t}\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

Example 4: Find a fundamental set of solutions of

$$x' = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix} x$$

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$$x' = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix} x$$

We compute the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{vmatrix}$$

$$= -16 - 4\lambda - 2\lambda^2 + \lambda^3 = (\lambda - 4)(\lambda + 2)^2.$$

Example 4: Find a fundamental set of solutions of

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$$= -16 - 4\lambda - 2\lambda^2 + \lambda^3 = (\lambda - 4)(\lambda + 2)^2.$$

Hence we have $\lambda_1 = 4$ with multiplicity one and $\lambda_2 = \lambda_3 = -2$ with multiplicity two.

Homogeneous Systems - Repeated Eigenvalues

From $\lambda_1 = 4$, we compute

$$(A - 4I) = \begin{pmatrix} -3-4 & 1 & -1 \\ -7 & 5-4 & -1 \\ -6 & 6 & -2-4 \end{pmatrix} = \begin{pmatrix} -7 & 1 & -1 \\ -7 & 1 & -1 \\ -6 & 6 & -6 \end{pmatrix}$$

Homogeneous Systems - Repeated Eigenvalues

From $\lambda_1 = 4$, we compute

$$(A - 4I) = \begin{pmatrix} -3-4 & 1 & -1 \\ -7 & 5-4 & -1 \\ -6 & 6 & -2-4 \end{pmatrix} = \begin{pmatrix} -7 & 1 & -1 \\ -7 & 1 & -1 \\ -6 & 6 & -6 \end{pmatrix}$$

Hence, solving $(A - 4I)x = 0$, we have

$$\left(\begin{array}{ccc|c} -7 & 1 & -1 & 0 \\ -7 & 1 & -1 & 0 \\ -6 & 6 & -6 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -6 & 6 & -6 & 0 \\ -7 & 1 & -1 & 0 \\ -7 & 1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

It follows that the eigenvector corresponding to $\lambda_1 = 4$ is $v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

and a solution of the system is

$$x_1 = e^{4t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Homogeneous Systems - Repeated Eigenvalues

From $\lambda_2 = -2$, we compute

$$(A + 2I) = \begin{pmatrix} -3+2 & 1 & -1 \\ -7 & 5+2 & -1 \\ -6 & 6 & -2+2 \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{pmatrix}$$

Homogeneous Systems - Repeated Eigenvalues

From $\lambda_2 = -2$, we compute

$$(A+2I) = \begin{pmatrix} -3+2 & 1 & -1 \\ -7 & 5+2 & -1 \\ -6 & 6 & -2+2 \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{pmatrix}$$

Hence, solving $(A+2I)x = 0$, we have

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ -7 & 7 & -1 & 0 \\ -6 & 6 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Homogeneous Systems - Repeated Eigenvalues

From $\lambda_2 = -2$, we compute

$$(A+2I) = \begin{pmatrix} -3+2 & 1 & -1 \\ -7 & 5+2 & -1 \\ -6 & 6 & -2+2 \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{pmatrix}$$

Hence, solving $(A+2I)x = 0$, we have

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ -7 & 7 & -1 & 0 \\ -6 & 6 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

It follows that the eigenvector corresponding to $\lambda_2 = -2$ is $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and a solution of the system is

$$x_2 = e^{-2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We need to find the third solution of the system, which has the form

$$x_3 = e^{-2t}w + te^{-2t}v_2$$

Homogeneous Systems - Repeated Eigenvalues

We need to find the third solution of the system, which has the form

$$x_3 = e^{-2t}w + te^{-2t}v_2$$

To find w, we solve

$$(A+2I)w = \begin{pmatrix} -1 & 1 & -1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Homogeneous Systems - Repeated Eigenvalues

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Hence

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & 1 \\ -7 & 7 & -1 & 1 \\ -6 & 6 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 0 & 6 & -6 \\ 0 & 0 & 7 & -7 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This gives $w_3 = -1$, $w_1 = w_2$.

Homogeneous Systems - Repeated Eigenvalues

We need to find the third solution of the system, which has the form

$$x_3 = e^{-2t} w + t e^{-2t} v_2$$

To find w, we solve

$$(A+2I)w = \begin{pmatrix} -1 & 1 & -1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Hence

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & 1 \\ -7 & 7 & -1 & 1 \\ -6 & 6 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 0 & 6 & -6 \\ 0 & 0 & 7 & -7 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This gives $w_3 = -1$, $w_1 = w_2$.

Hence choosing $w_1 = 1$ we have $w = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

Homogeneous Systems - Repeated Eigenvalues

In conclusion, we have the Fundamental Set:

$$x_1 = e^{4t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad x_2 = e^{-2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad x_3 = e^{-2t}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + te^{-2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Homogeneous Systems - Repeated Eigenvalues

In conclusion, we have the Fundamental Set:

$$x_1 = e^{4t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad x_2 = e^{-2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad x_3 = e^{-2t}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + te^{-2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

General solution:

$$x = C_1 e^{4t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_3\left[e^{-2t}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + te^{-2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right]$$

Solution of Homogeneous Systems with 3 repeated eigenvalues

Given the differential system

$$x' = Ax.$$

Suppose that λ is an eigenvalue of A of multiplicity 3. Then exactly one of the following three cases holds:

Solution of Homogeneous Systems with 3 repeated eigenvalues

Given the differential system

$$x' = Ax.$$

Suppose that λ is an eigenvalue of A of multiplicity 3. Then exactly one of the following three cases holds:

1. λ has three linearly independent eigenvectors v_1, v_2, v_3. Then three linearly independent solution vectors of the system corresponding to λ are:

$$x_1(t) = e^{\lambda t} v_1, \quad x_2(t) = e^{\lambda t} v_2, \quad x_3(t) = e^{\lambda t} v_3.$$

Homogeneous Systems - Repeated Eigenvalues

Solution of Homogeneous Systems with 3 repeated eigenvalues

2. λ has two lin. independent eigenvectors v_1, v_2. Then three lin. independent solutions of the system corresponding to λ are:

$$x_1(t) = e^{\lambda t} v_1, \quad x_2(t) = e^{\lambda t} v_2 \text{ and } x_3(t) = e^{\lambda t} w + t e^{\lambda t} v$$

where v is an eigenvector corresponding to λ and $(A - \lambda I)w = v$; that is: $(A - \lambda I)^2 w = 0$.

Homogeneous Systems - Repeated Eigenvalues

Solution of Homogeneous Systems with 3 repeated eigenvalues

2. λ has two lin. independent eigenvectors $v_1,\ v_2$. Then three lin. independent solutions of the system corresponding to λ are:

$$x_1(t) = e^{\lambda t} v_1, \quad x_2(t) = e^{\lambda t} v_2 \ \text{ and } \ x_3(t) = e^{\lambda t} w + t e^{\lambda t} v$$

   where v is an eigenvector corresponding to λ and $(A - \lambda I)w = v$; that is: $(A - \lambda I)^2 w = 0$.

3. λ has only one (independent) eigenvector v. Then three linearly independent solutions of the system have the form:

$$x_1 = e^{\lambda t} v, \quad x_2 = e^{\lambda t} w + t e^{\lambda t} v, \ \text{ and } \ x_3(t) = e^{\lambda t} z + t e^{\lambda t} w + t^2 e^{\lambda t} v$$

   where $(A - \lambda I)z = w$ & $(A - \lambda I)w = v$;
   that is, $(A - \lambda I)^3 z = 0$ & $(A - \lambda I)^2 w = 0$

Example:

$$y''' - 6y'' + 12y' - 8y = 0$$

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We have the characteristic equation: $(r-2)^3 = 0$

We have the characteristic roots: $r_1 = r_2 = r_3 = 2$

Hence we have the fundamental set:

$$\left\{e^{2t},\ te^{2t},\ t^2e^{2t}\right\}$$

Homogeneous Systems - 3 repeated Eigenvalues

Corresponding system:

$$x' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -12 & 6 \end{pmatrix} x$$

Homogeneous Systems - 3 repeated Eigenvalues

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$$x' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -12 & 6 \end{pmatrix} x$$

We compute the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 8 & -12 & 6-\lambda \end{vmatrix}$$

$$= 8 - 12\lambda + 6\lambda^2 - \lambda^3 = (\lambda - 2)^3.$$

Hence we have one eigenvalue $\lambda_1 = \lambda_2 = \lambda_3 = 2$ with multiplicity 3.

Homogeneous Systems - 3 repeated Eigenvalues

To find the eigenvector of $\lambda_1 = 2$, we compute

$$(A - 2I) = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 8 & -12 & 6-2 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 8 & -12 & 4 \end{pmatrix}$$

Homogeneous Systems - 3 repeated Eigenvalues

To find the eigenvector of $\lambda_1 = 2$, we compute

$$(A - 2I) = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 8 & -12 & 6-2 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 8 & -12 & 4 \end{pmatrix}$$

Hence, solving $(A - 2I)x = 0$, we have

$$\left(\begin{array}{rrr|r} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 8 & -12 & 4 & 0 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 1 & -1/2 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -8 & 4 & 0 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Homogeneous Systems - 3 repeated Eigenvalues

To find the eigenvector of $\lambda_1 = 2$, we compute

$$(A - 2I) = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 8 & -12 & 6-2 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 8 & -12 & 4 \end{pmatrix}$$

Hence, solving $(A - 2I)x = 0$, we have

$$\left(\begin{array}{rrr|r} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 8 & -12 & 4 & 0 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & -1/2 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -8 & 4 & 0 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

It follows that the eigenvector corresponding to $\lambda_1 = 1$ is $v = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

and a solution of the system is

$$x_1 = e^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

Homogeneous Systems - 3 repeated Eigenvalues

Next, we look for the second solution of the system, which has the form

$$x_2 = e^{2t}w + te^{2t}v$$

Homogeneous Systems - 3 repeated Eigenvalues

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$$x_2 = e^{2t}w + te^{2t}v$$

To find w, we solve

$$(A - 2I)w = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 8 & -12 & 4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

Homogeneous Systems - 3 repeated Eigenvalues

Next, we look for the second solution of the system, which has the form

$$x_2 = e^{2t}w + te^{2t}v$$

To find w, we solve

$$(A-2I)w = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 8 & -12 & 4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

Hence

$$\left(\begin{array}{rrr|r} -2 & 1 & 0 & 1 \\ 0 & -2 & 1 & 2 \\ 8 & -12 & 4 & 4 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -2 & 1 & 2 \\ 0 & -8 & 4 & 8 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We find the generalized eigenvector $w = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$

Homogeneous Systems - 3 repeated Eigenvalues

Next, we look for the third solution of the system, which has the form

$$x_3 = e^{2t}z + te^{2t}w + t^2e^{2t}v$$

Homogeneous Systems - 3 repeated Eigenvalues

Next, we look for the third solution of the system, which has the form

$$x_3 = e^{2t}z + te^{2t}w + t^2e^{2t}v$$

To find z, we solve

$$(A-2I)z = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 8 & -12 & 4 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$$

Hence

$$\left(\begin{array}{rrr|r} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 \\ 8 & -12 & 4 & 4 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & -8 & 4 & 4 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We find the generalized eigenvector $z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Homogeneous Systems - 3 repeated Eigenvalues

Fundamental set:

$$x_1 = e^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \quad x_2 = e^{2t} \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} + te^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix},$$

$$x_3 = e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + te^{2t} \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} + t^2 e^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

Non-Homogeneous Systems

The treatment in this topic parallels exactly the treatment of linear nonhomogeneous equations discussed in Section 3.

We are going to provide just some general observations without getting into details.

Non-Homogeneous Systems

Recall that a general nonhomogeneous system of first-order linear differential equations has the form

$$\begin{array}{rcl} x_1' & = & a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + b_1(t) \\ x_2' & = & a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + b_2(t) \\ \vdots & & \qquad\qquad\qquad \vdots \\ x_n' & = & a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + b_n(t) \end{array}$$

Non-Homogeneous Systems

Using the notation

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

and

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad b(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix},$$

the first-order linear differential system can be written in the **vector-matrix form**

$$x' = A(t)\,x + b(t). \tag{S}$$

Non-Homogeneous Systems

The following result generalizes a property of standard linear differential equations

Non-Homogeneous Systems

The following result generalizes a property of standard linear differential equations

Theorem

If z_1 and z_2 are solutions of the nonhomogeneous system (S), then $x = z_1 - z_2$ is a solution of the corresponding homogeneous system (H).

Non-Homogeneous Systems

The following result generalizes a property of standard linear differential equations

Theorem

If z_1 and z_2 are solutions of the nonhomogeneous system (S), then $x = z_1 - z_2$ is a solution of the corresponding homogeneous system (H).

Proof: Since $\mathbf{z}_1$ and $\mathbf{z}_2$ are solutions of (N),

$$\mathbf{z}_1'(t) = A(t)\mathbf{z}_1(t) + \mathbf{b}(t) \quad \text{and} \quad \mathbf{z}_2'(t) = A(t)\mathbf{z}_2(t) + \mathbf{b}(t)).$$

Let $\mathbf{x}(t) = \mathbf{z}_1(t) - \mathbf{z}_2(t)$. Then

$$\mathbf{x}'(t) = \mathbf{z}_1'(t) - \mathbf{z}_2'(t) = [A(t)\mathbf{z}_1(t) + \mathbf{b}(t)] - [A(t)\mathbf{z}_2(t) + \mathbf{b}(t)]$$

$$= A(t)\,[\mathbf{z}_1(t) - \mathbf{z}_2(t)] = A(t)\mathbf{x}(t).$$

Thus, $\mathbf{x}(t) = \mathbf{z}_1(t) - \mathbf{z}_2(t)$ is a solution of (H). ■

Non-Homogeneous Systems

The following result also generalizes a fundamental result of standard linear differential equations

Non-Homogeneous Systems

The following result also generalizes a fundamental result of standard linear differential equations

Theorem

Let $x_1, x_2, \ldots, x_n$ be a fundamental set of solutions the reduced system (H) and z be a particular solution of (S). If u is a solution of (S), then there are real numbers $C_1, C_2, \ldots, C_n$ such that

$$u(t) = C_1 x_1(t) + C_2 x_2(t) + \cdots + C_n x_n(t) + z(t).$$

Proof: Let $\mathbf{u} = \mathbf{u}(t)$ be *any* solution of (N). By Theorem 1, $\mathbf{u}(t) - \mathbf{z}(t)$ is a solution of the reduced system (H). Since $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ are n linearly independent solutions of (H), there exist constants $c_1, c_2, \ldots, c_n$ such that

$$\mathbf{u}(t) - \mathbf{z}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t).$$

Therefore

$$\mathbf{u}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) + \mathbf{z}(t). \qquad \blacksquare$$

Non-Homogeneous Systems

By the last theorem, if $x_1,\ x_2,\ \ldots,\ x_n$ are linearly independent solution of (H) and z is a particular solution of (S), then

$$u(t) = C_1x_1(t) + C_2x_2(t) + \cdots + C_nx_n(t) + z(t)$$

is the **general solution** of (S), in the sense that any solution of (S) is of this form.

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To find a particular solution, one can use the method of Variation of Parameters.