3.1 Elementary Matrix Operations and Elementary Matrix

- Elementary Matrix Operations
- Solving a System by Row Eliminations: Example
- Elementary Matrix
- Multiplication by Elementary Matrices
- Properties of Elementary Operations
- Inverses of Elementary Matrices

Elementary Matrix Operations

Definition (Elementary Matrix Operations)

Elementary row/column operations on an $m \times n$ matrix A:

1. *(Interchange)* interchanging any two rows/columns
2. *(Scaling)* multiplying any row/column by nonzero scalar
3. *(Replacement)* adding any scalar multiple of a row/column to another row/column

Row Equivalent Matrices

Two matrices where one matrix can be transformed into the other matrix by a sequence of elementary row operations.

Fact about Row Equivalence

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Solving a System by Row Eliminations: Example

Example (Row Eliminations to a Triangular Form)

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Solving a System by Row Eliminations: Example (cont.)

Example (Row Eliminations to a Diagonal Form)

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & -3 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{array}{rrrcr} x_1 & & & = & 29 \\ & x_2 & & = & 16 \\ & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Solution: $(29, 16, 3)$

Elementary Matrix

Definition

An $n \times n$ elementary matrix is obtained by performing an elementary operation on I_n. It is of type 1, 2, or 3, depending on which elementary operation was performed.

Example

Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

E_1, E_2, and E_3 are elementary matrices. Why?

Multiplication by Elementary Matrices

Observe the following products and describe how these products can be obtained by elementary row operations on A.

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{bmatrix}$$

$$E_2A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$$

$$E_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 3a+g & 3b+h & 3c+i \end{bmatrix}$$

If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operations on I_m.

Properties of Elementary Operations

Theorem (3.1)

Let $A \in M_{m\times n}(F)$, and B obtained from an elementary row (or column) operation on A. Then there exists an $m \times m$ (or $n \times n$) elementary matrix E s.t. $B = EA$ (or $B = AE$). This E is obtained by performing the same operation on I_m (or I_n). Conversely, for elementary E, then EA (or AE) is obtained by performing the same operation of A as that which produces E from I_m (or I_n).

Example: Row Eliminations to a Triangular Form - Step 1

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} = A$$

$$\Downarrow \quad E_1$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix} = A_1$$

$$A_1 = E_1 A, \qquad E_1 = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

Example: Row Eliminations to a Triangular Form - Step 2

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix} = A_1$$

$$\Downarrow \quad E_2$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix} = A_2$$

$$A_2 = E_2 A_1, \qquad E_2 = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Example: Row Eliminations to a Triangular Form - Step 3

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix} = A_2$$

$$\Downarrow \quad E_3$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} = A_3$$

$$A_3 = E_3 A_2, \qquad E_3 = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Example: Row Eliminations to a Diagonal Form - Step 4

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} = A_3$$

$$\Downarrow \quad E_4$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & -3 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix} = A_4$$

$$A_4 = E_4 A_3, \qquad E_4 = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Example: Row Eliminations to a Diagonal Form - Step 5

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & -3 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix} = A_4$$

$$\Downarrow \quad E_5$$

$$\begin{array}{rcrcrcr} x_1 & & & & & = & 29 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix} = A_5$$

$$A_5 = E_5 A_4, \qquad E_5 = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Inverses of Elementary Matrices

Theorem (3.2)

Elementary matrices are invertible, and the inverse is an elementary matrix of the same type.

Elementary matrices are *invertible* because row operations are *inversible*. To determine the inverse of an elementary matrix E, determine the elementary row operation needed to transform E back into I and apply this operation to I to find the inverse.

Example

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad E_3^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Inverses of Elementary Matrices: Examples

Example

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_1^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Example

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad E_2^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

3.2 The Rank of a Matrix and Matrix Inverses

- The Rank of a Matrix
  - Defintion
  - Properties of the Rank of a Matrix
  - Determining the Rank of a Matrix
  - Rank of Matrix Products
- The Matrix Inverses

The Rank of a Matrix

Definition (The Rank of a Matrix)

The rank of a matrix $A \in M_{m\times n}(F)$ is the rank of the linear transformation $L_A : F^n \to F^m$.

$$\mathrm{rank}(A) = \mathrm{rank}(L_A) = \dim(R(L_A))$$

An $n \times n$ matrix is invertible if and only if its rank is n.

The Rank of a Matrix

Theorem (3.3)

Let $T : V \to W$ be linear between finite-dimensional V, W with ordered bases β, γ. Then

$$\mathrm{rank}(T) = \mathrm{rank}([T]_\beta^\gamma).$$

$$\mathrm{rank}(T) = \mathrm{rank}(L_A), \mathrm{nullity}(T) = \mathrm{nullity}(L_A), \quad \text{with } A = [T]_\beta^\gamma$$

Properties of the Rank of a Matrix

Theorem (3.4)

Let A be $m \times n$, and P, Q invertible of sizes $m \times m$, $n \times n$. Then

(a) $\mathrm{rank}(AQ) = \mathrm{rank}(A)$

(b) $\mathrm{rank}(PA) = \mathrm{rank}(A)$

(c) $\mathrm{rank}(PAQ) = \mathrm{rank}(A)$

(a) Note

$$R(L_{AQ}) = R(L_A L_Q) = L_A L_Q(F^n) = L_A(L_Q(F^n)) = L_A(F^n) = R(L_A).$$

Then

$$\mathrm{rank}(AQ) = \dim(R(L_{AQ})) = \dim(R(L_A)) = \mathrm{rank}(A).$$

Corollary

Elementary row/column operations are rank-preserving.

Determining the Rank of a Matrix

Theorem (3.5)

$\mathrm{rank}(A)$ *is the maximum number of linearly independent columns of A, that is, the dimension of the subspace generated by its columns.*

Note

$$R(L_A) = \mathrm{span}(\{L_A(e_1), \cdots, L_A(e_n)\}) = \mathrm{span}(\{a_1, \cdots, a_n\})$$

where $L_A(e_j) = Ae_j = a_j$, with a_j the jth column of A. Then

$$\mathrm{rank}(A) = \mathrm{rank}(L_A) = \dim(R(L_A)) = \dim(\mathrm{span}(\{a_1, \cdots, a_n\}))$$

Determining the Rank of a Matrix

Elementary row/column operations are rank-preserving.

Example (Row Reduction to Echelon Form)

$$A = \begin{bmatrix} -1 & 2 & 3 & 6 \\ 2 & -5 & -6 & -12 \\ 1 & -3 & -3 & -6 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & 6 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{rank}(A) = ?$$

Determining the Rank of a Matrix (cont.)

Theorem (3.6)

Let A be $m \times n$ with $\operatorname{rank}(A) = r$. Then $r \leq m$, $r \leq n$, and by finite number of elementary row/column operations A can be transformed into

$$D = \begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}$$

where O_1, O_2, O_3 are zero matrices, that is, $D_{ii} = 1$ for $i \leq r$ and $D_{ij} = 0$ otherwise.

Elementary row/column operations are rank-preserving.

$$A = \begin{bmatrix} -1 & 2 & 3 & 6 \\ 2 & -5 & -6 & -12 \\ 1 & -3 & -3 & -6 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & 6 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\operatorname{rank}(A) = r = 2$$

Determining the Rank of a Matrix (cont.)

Corollary 1

Let A be $m \times n$ of rank r. Then there exists invertible B, C of sizes $m \times m$, $n \times n$ such that

$$D = BAC = \begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}$$

$$BAC = [B_{ij}?] \begin{bmatrix} -1 & 2 & 3 & 6 \\ 2 & -5 & -6 & -12 \\ 1 & -3 & -3 & -6 \end{bmatrix} [C_{lk}?] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = D$$

$$B = \begin{bmatrix} \quad \end{bmatrix}, \quad C = \begin{bmatrix} \quad \end{bmatrix}$$

Determining the Rank of a Matrix (cont.)

Corollary 2

Let A be $m \times n$, then

(a) $\mathrm{rank}(A^t) = \mathrm{rank}(A)$

(b) $\mathrm{rank}(A)$ is the maximum number of linearly independent rows, that is, the dimension of the subspace generated by its rows.

(c) The rows and columns of A generate subspaces of the same dimension, namely $\mathrm{rank}(A)$

Corollary 3

Every invertible matrix is a product of elementary matrices.

Matrix Inverses as Products of Elementary Matrices

Example

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$. Then

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_2 (E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E_3 (E_2 E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix Inverses as Products of Elementary Matrices (cont.)

Example (cont.)

So

$$\boxed{E_3 E_2 E_1 A = I_3}.$$

Then multiplying on the right by A^{-1}, we get

$$E_3 E_2 E_1 A\_\_\_\_\_\_ = I_3\_\_\_\_\_\_.$$

So

$$\boxed{E_3 E_2 E_1 I_3 = A^{-1}}$$

Rank of Matrix Products

Theorem (3.7)

Let $T : V \to W$ and $U : W \to Z$ be linear on finite-dimensional V, W, Z. Let A, B be matrices such that AB is defined. Then

(a) $\operatorname{rank}(UT) \leq \operatorname{rank}(U)$

(b) $\operatorname{rank}(UT) \leq \operatorname{rank}(T)$

(c) $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$

(d) $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$

The Inverse of a Matrix

Definition

Let A, B be $m \times n$, $m \times p$ matrices. The augmented matrix $(A|B)$ is the $m \times (n+p)$ matrix $(A\,B)$.

If A is invertible $n \times n$, then $(A|I_n)$ can be transformed into $(I_n|A^{-1})$ by finite number of elementary row operations.

If A is invertible $n \times n$ and $(A|I_n)$ is transformed into $(I_n|B)$ by finite number of elementary row operations, then $B = A^{-1}$.

If A is non-invertible $n \times n$, then any attempt to transform $(A|I_n)$ into $(I_n|B)$ produces a row whose first n entries are zero.

The Inverses of Matrix: Example

Example

Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

Solution:

$$[A \quad I] = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{bmatrix}$$

So $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$

3.3 Systems of Linear Equations – Theoretical Aspects

- Systems of Linear Equations
- Solution Sets: Homogeneous System
- Solution Sets: Nonhomogeneous System
- Invertibility
- Consistency

Systems of Linear Equations

System of m linear equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

or

$$Ax = b$$

with coefficient matrix A and vectors x, b:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Solution Sets

- A solution to the system $Ax = b$:

$$s = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} \in F^n \quad \text{such that } As = b.$$

- The solution set of the system: The set of all solutions
- Consistent system: Nonempty solution set
- Inconsistent system: Empty solution set

Solution Sets: Homogeneous System

Definition

$Ax = b$ is homogeneous if $b = 0$, otherwise nonhomogeneous.

Theorem (3.8)

Let $Ax = 0$ be a homogeneous system of m equations in n unknowns. The set of all solutions to $Ax = 0$ is $K = N(L_A)$, which is a subspace of F^n of dimension $n - \mathrm{rank}(L_A) = n - \mathrm{rank}(A)$.

Homogeneous System: Trivial Solutions

Example

$$\begin{array}{rcrcl} x_1 & + & 10x_2 & = & 0 \\ 2x_1 & + & 20x_2 & = & 0 \end{array}$$

Corresponding matrix equation $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 10 \\ 2 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Trivial solution: $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or $\mathbf{x} = \mathbf{0}$

Homogeneous System: Nontrivial Solutions

The homogeneous system $A\mathbf{x} = \mathbf{0}$ *always* has the **trivial solution**, $\mathbf{x} = \mathbf{0}$.

Nontrivial Solution

Nonzero vector solutions are called **nontrivial solutions.**

Example (cont.)

Do **nontrivial** solutions exist?

$$\begin{bmatrix} 1 & 10 & 0 \\ 2 & 20 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Consistent system with a free variable has infinitely many solutions.

A homogeneous equation $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions if and only if the system of equations has

__________________________________.

Homogeneous System: Example 1

Example (1)

Determine if the following homogeneous system has nontrivial solutions and then describe the solution set.

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & - & 6x_3 & = & 0 \\ 4x_1 & + & 8x_2 & - & 10x_3 & = & 0 \end{array}$$

Solution: There is at least one free variable (why?)
$\Longrightarrow$ nontrivial solutions exist

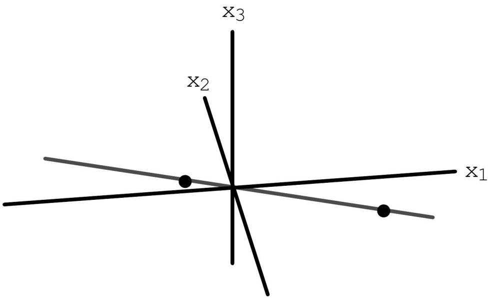
$$\begin{bmatrix} 2 & 4 & -6 & 0 \\ 4 & 8 & -10 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 4 & 8 & -10 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Longrightarrow \begin{array}{ll} x_1 = & \\ x_2 & \text{is free} \\ x_3 = & \end{array}$$

Homogeneous System: Example 1 (cont.)

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} = \_\_\_\_ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = x_2\mathbf{v}$$

Graphical representation:



solution set = span$\{\mathbf{v}\}$ = line through $\mathbf{0}$ in $\mathbf{R}^3$

Homogeneous System: Non Trivial Solutions

Corollary

If $m < n$, the system $Ax = 0$ has a nonzero solution.

Solution Sets: Nonhomogeneous System

Theorem (3.9)

Let K be the solution set of $Ax = b$, and let K_H be the solution set of the corresponding homogeneous system $Ax = 0$. Then for any solution s to $Ax = b$:

$$K = \{s\} + K_H = \{s + k : k \in K_H\}.$$

Nonhomogeneous System: Example 2

Example (2)

Describe the solution set of

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & - & 6x_3 & = & 0 \\ 4x_1 & + & 8x_2 & - & 10x_3 & = & 4 \end{array}$$

(same left side as in the previous example)

Solution:

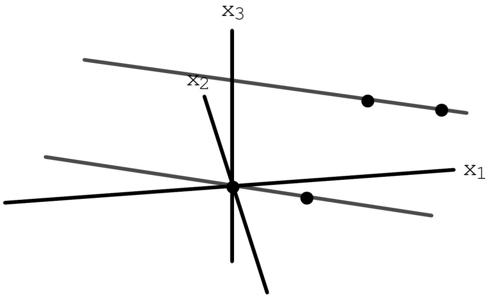
$$\begin{bmatrix} 2 & 4 & -6 & 0 \\ 4 & 8 & -10 & 4 \end{bmatrix} \quad \text{row reduces to} \quad \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

Nonhomogeneous System: Example 2 (cont.)

$$\mathbf{x} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{p} + x_2\mathbf{v}$$

Graphical representation:



Parallel solution sets of $A\mathbf{x} = \mathbf{0}$ & $A\mathbf{x} = \mathbf{b}$

Nonhomogeneous System: Recap of Previous Two Examples

Example (1. Solution of $A\mathbf{x} = \mathbf{0}$)

$$\mathbf{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = x_2\mathbf{v}$$

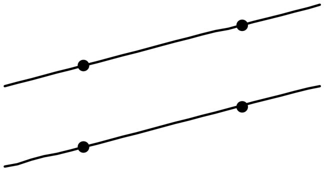
$\mathbf{x} = x_2\mathbf{v}$ = parametric equation of line passing through $\mathbf{0}$ and $\mathbf{v}$

Example (2. Solution of $A\mathbf{x} = \mathbf{b}$)

$$\mathbf{x} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{p} + x_2\mathbf{v}$$

$\mathbf{x} = \mathbf{p} + x_2\mathbf{v}$ = parametric equation of line passing through $\mathbf{p}$ parallel to $\mathbf{v}$

Nonhomogeneous System



Parallel solution sets of
$A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Nonhomogeneous System: Example

Example

Describe the solution set of $2x_1 - 4x_2 - 4x_3 = 0$; compare it to the solution set $2x_1 - 4x_2 - 4x_3 = 6$.

Solution: Corresponding augmented matrix to $2x_1 - 4x_2 - 4x_3 = 0$:

$$\begin{bmatrix} 2 & -4 & -4 & 0 \end{bmatrix} \sim \qquad \text{(fill-in)}$$

Vector form of the solution:

$$\mathbf{v} = \begin{bmatrix} 2x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \_\_\_\_ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

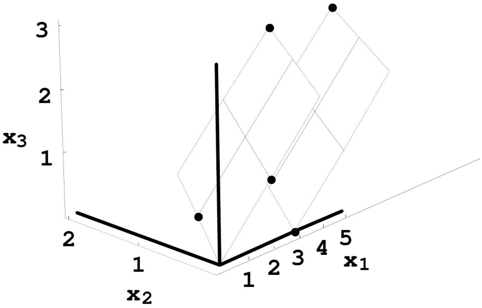
Corresponding augmented matrix to $2x_1 - 4x_2 - 4x_3 = 6$:

$$\begin{bmatrix} 2 & -4 & -4 & 6 \end{bmatrix} \sim \qquad \text{(fill -in)}$$

Nonhomogeneous System: Example (cont.)

Vector form of the solution:

$$\mathbf{v} = \begin{bmatrix} 3 + 2x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} + \text{\_\_\_\_} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \text{\_\_\_\_} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$



Parallel Solution Sets of $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$

Invertibility

Theorem (3.10)

If A is invertible then the system $Ax = b$ has exactly one solution $x = A^{-1}b$. Conversely, if the system has exactly one solution then A is invertible.

Consistency

Theorem (3.11)

The system $Ax = b$ is consistent if and only if

$$\mathrm{rank}(A) = \mathrm{rank}(A|b)$$

3.4 Systems of Linear Equations – Computational Aspects

- Equivalent Systems
- Reduced Row Echelon Form
- Gaussian Elimination
- General Solutions
- Interpretation of the Reduced Row Echelon Form

Equivalent Systems

Definition

Two systems of linear equations are called **equivalent** if they have the same solution set.

Theorem (3.13)

For $m \times n$ linear system $Ax = b$ and invertible $m \times m$ matrix C, the system $(CA)x = Cb$ is equivalent to $Ax = b$.

Equivalent Systems

Corollary

For linear system $Ax = b$, if $(A'|b')$ is obtained from $(A|b)$ by a finite number of elementary row operations, then $A'x = b'$ is equivalent to the original system.

Reduced Row Echelon Form

Definition

A matrix is in reduced row echelon form if:

(a) Any row containing a nonzero entry precedes any row in which all the entries are zero

(b) The first nonzero entry in each row is the only nonzero entry in its column

(c) The first nonzero entry in each row is 1 and it occurs in a column right of the first nonzero entry in the preceding row.

Example

$$\begin{pmatrix} 1 & 0 & x & 0 & x & 0 & x & x \\ & 1 & x & 0 & x & 0 & x & x \\ & & & 1 & x & 0 & x & x \\ & & & & & 1 & x & x \\ \\ \\ \end{pmatrix}$$

Gaussian Elimination

Definition (Gaussian Elimination)

Reducing an augmented matrix to reduced row echelon form:

- In the **forward pass**, the matrix is transformed into upper triangular form where first nonzero entry of each row is 1, in a column to the right of the first nonzero entry of preceding rows.
- In the **backward pass** or **back-substitution**, the matrix is transformed into reduced row echelon form by making the first nonzero entry of each row the only nonzero entry of its column.

Pivots

Important Terms

- **pivot position:** a position of a leading entry in an echelon form of the matrix.
- **pivot:** a nonzero number that either is used in a pivot position to create 0's or is changed into a leading 1, which in turn is used to create 0's.
- **pivot column:** a column that contains a pivot position.

Reduced Echelon Form: Examples

Example (Row reduce to echelon form and locate the pivots)

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Solution

pivot ↙

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

↑ pivot column

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Possible Pivots:

Reduced Echelon Form: Examples (cont.)

Example (Row reduce to echelon form (cont.))

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Original Matrix: } \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

$$\begin{matrix} & \uparrow & \uparrow & & \uparrow \\ \text{pivot columns:} & 1 & 2 & & 4 \end{matrix}$$

Note

There is no more than one pivot in any row. There is no more than one pivot in any column.

Reduced Echelon Form: Examples (cont.)

Example (Row reduce to echelon form and then to REF)

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Reduced Echelon Form: Examples (cont.)

Example (Row reduce to echelon form and then to REF (cont.))

Cover the top row and look at the remaining two rows for the left-most nonzero column.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \text{(echelon form)}$$

Reduced Echelon Form: Examples (cont.)

Example (Row reduce to echelon form and then to REF (cont.))

Final step to create the reduced echelon form:

Beginning with the rightmost leading entry, and working upwards to the left, create zeros above each leading entry and scale rows to transform each leading entry into 1.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination

Theorem (3.14)

Gaussian elimination transforms any matrix into its reduced row echelon form.

Solutions of Linear Systems

Important Terms

- **basic variable:** any variable that corresponds to a pivot column in the augmented matrix of a system.
- **free variable:** all nonbasic variables.

Example (Solutions of Linear Systems)

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\begin{aligned} x_1 \quad +6x_2 \qquad\quad +3x_4 \qquad &= 0 \\ x_3 \quad -8x_4 \qquad &= 5 \\ x_5 \quad &= 7 \end{aligned}$$

pivot columns:
basic variables:
free variables:

Solutions of Linear Systems (cont.)

Final Step in Solving a Consistent Linear System

After the augmented matrix is in **reduced** echelon form and the system is written down as a set of equations, *Solve each equation for the basic variable in terms of the free variables (if any) in the equation.*

Example (General Solutions of Linear Systems)

$$\begin{array}{cccccc} x_1 & +6x_2 & & +3x_4 & & = 0 \\ & & x_3 & -8x_4 & & = 5 \\ & & & & x_5 & = 7 \end{array}$$

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 8x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$$

(general solution)

Warning

Use only the reduced echelon form to solve a system.

General Solutions of Linear Systems

General Solution

The **general solution** of the system provides a parametric description of the solution set. (The free variables act as parameters.)

Example (General Solutions of Linear Systems (cont.))

$$
\begin{aligned}
x_1 &= -6x_2 - 3x_4 \\
x_2 &\text{ is free} \\
x_3 &= 5 + 8x_4 \\
x_4 &\text{ is free} \\
x_5 &= 7
\end{aligned}
$$

The above system has **infinitely many solutions.** Why?

General Solutions

Theorem (3.15)

Let $Ax = b$ be a system of r nonzero equations in n unknowns. Suppose rank$(A) = \mathrm{rank}(A|b)$ and that $(A|b)$ is in reduced row echelon form. Then

(a) *rank$(A) = r$.*

(b) *If the general solution is of the form*

$$s = s_0 + t_1 u_1 + t_2 u_2 + \cdots + t_{n-r} u_{n-r}$$

then $\{u_1, u_2, \cdots, u_{n-r}\}$ is a basis for the solution set of the corresponding homogeneous system, and s_0 is a solution to the original system.

Interpretation of the Reduced Row Echelon Form

Theorem (3.16)

Let A be an $m \times n$ matrix of rank $r > 0$ and B the reduced row echelon form of A. Then

(a) *The number of nonzero rows in B is r.*

(b) *For each $i = 1, \cdots, r$, there is a column b_{j_i} of B s.t. $b_{j_i} = e_i$*

(c) *The columns of A numbered $j_1, \cdots, j_r$ are linearly independent*

(d) *For each $k = 1, \cdots, n$, if column k of B is $d_1 e_1 + \cdots + d_r e_r$ then column k of A is $d_1 a_{j_1} + \cdots + d_r a_{j_r}$*

Interpretation of the Reduced Row Echelon Form

Corollary

The reduced row echelon form of a matrix is unique.