Mechanisms of Symmetry Creation

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1 Introduction

Numerical simulation in [1] indicates that symmetry increasing bifurcations of chaotic attractors occur with great frequency in the dynamics of symmetric mappings. The pictures in [1], [3], [5] also demonstrate that an unexpected kind of pattern formation occurs in symmetric chaotic dynamics where an order based on symmetry is forced on the randomness that is related to chaotic dynamics. Inspection of various numerical simulations in the literature also show that symmetry increasing bifurcations have been observed in ODEs (the *Lorenz equation* in [8]) and in Galerkin approximations of PDEs (the *Ginzburg-Landau equation* in [6]).

In this paper we make a more detailed numerical analysis of how symmetry creation can occur and how it may be related to pattern formation as the term is used in Physics and Engineering. In applications, the fundamental question concerns how the symmetry of an attractor in phase space manifests itself in physical space. The important point to be noted here is that the symmetry of the attractor in phase space is only "on average". One must either iterate the mapping a relatively large number of times, or analogously integrate the differential equation for a relatively long time in order to see that symmetry. It follows that if the symmetry of the attractor is to be seen in physical variables (as a pattern) it must be seen through some averaged quantity.

In Section 2 we indicate by example how averaged quantities can undergo symmetry creation. We consider both the *Brusselator* and the *Ginzburg-Landau equation*. Although these equations each have only a single reflectional symmetry, the types of symmetry increasing they exhibit are quite different. In Section 3 we illustrate these differences by considering the discrete dynamics of odd maps on the line. In this section we also consider how the parameter values where symmetry creation occurs may be computed by methods other than direct simulation. These techniques are based on theoretical results in [2].

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Generally speaking we think of symmetry creation in systems with finite symmetry as occurring through the "collision" of symmetry related conjugate attractors. As we show in Section 3 this is not always the basis for symmetry creation — but collisions do occur frequently and it is a useful way of thinking. In systems with continuous symmetry there is another method by which symmetry creation can occur — drifting along group orbits. We illustrate this phenomenon in Section 4 by using an example of a mapping on $\mathbb{R}^4$ having $\mathbf{O}(2)$ symmetry. We discuss briefly why this type of symmetry creation may be the method by which turbulent wavy vortices turn into turbulent Taylor vortices in the Couette-Taylor experiment.

2 Symmetry creation in PDEs

In this section we document the occurrence of symmetry increasing bifurcations in PDEs by considering two examples: the *Brusselator* and the *Ginzburg-Landau equation*. In each case we detect these bifurcations through the use of appropriately defined time averaged quantities.

(a) The Brusselator

We consider the following system of reaction-diffusion equations on the interval $[0,1]$

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{D_1}{\lambda^2}\frac{\partial^2 u}{\partial x^2} + u^2 v - (B+1)u + A \\ \frac{\partial v}{\partial t} &= \frac{D_2}{\lambda^2}\frac{\partial^2 v}{\partial x^2} - u^2 v + Bu\,. \end{aligned} \tag{2.1}$$

This system is known as the *Brusselator*, in which u, v, A and B represent chemical concentrations and D_1, D_2 are diffusion constants. In this simplified model A and B are assumed to be constant in both space and time while u and v depend on x and t. The parameter λ is a characteristic dimension of the system and we shall treat λ as the bifurcation parameter. We consider (2.1) on the interval $[0,1]$ subject to the Dirichlet boundary conditions

$$\begin{aligned} u(0,t) &= u(1,t) = A, \\ v(0,t) &= v(1,t) = B/A. \end{aligned}$$

This Dirichlet problem possesses a reflectional symmetry given by

$$\kappa(u(x,t), v(x,t)) = (u(1-x,t), v(1-x,t)).$$

Holodniok et. al. [4] have considered this model in detail and found multifrequency motions using numerical simulation. Presuming that chaotic dynamics will occur near multifrequency motion we follow [4] and set

$$A = 2, \quad B = 5.45, \quad D_1 = 0.008, \quad D_2 = 0.004.$$

As we mentioned in the introduction the symmetry of a chaotic attractor is expected to be a *symmetry on average*. Therefore it can only be seen by averaging over a long time. But what quantity should be averaged?

The averaged quantity that we use to detect symmetry creation may be interpreted through the following hypothetical story. Suppose that the radical whose concentration is being measured by u is an acid and that u is actually measuring the concentration of this acid along the bottom of a (two dimensional) rectangular container. Suppose further that this acid can eat away or etch the bottom of this container. It would then be reasonable that the rate of etching — at any point along the bottom — would be proportional to the concentration u and that the total amount of the bottom that would be etched is proportional to

$$F_u(x) = \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T u(x,\tau)d\tau\right) . \tag{2.2}$$

Suppose that the attractor A corresponding to the solution $u(x,t)$ is chaotic and symmetric. We expect that u would sample the whole attractor in phase space rather quickly and that the time average (2.2) would be equal to a (weighted) space average over the attractor in phase space — though the rigorous proof of this point would require an ergodic type theorem to be valid. Presuming this we would expect that (2.2) would be symmetric under $x \to 1-x$ if the attractor is symmetric. Moreover, generically, we would expect (2.2) to be asymmetric should A be asymmetric.

The graphs of F_u both before and after a symmetry increasing bifurcation are shown in Figure 1. In both cases $T = 20000$ was chosen for the numerical computations. In Figure 2 we indicate why we believe that this symmetry creation is caused by collision of conjugate attractors. If such a collision were to occur, the difference in phase space between the union of A and its conjugate attractor before collision would be approximately equal to A after collision. Applying the presumed ergodic theorem mentioned previously we would expect the average

$$\frac{1}{2}\left(F_u(x) + F_u(1-x)\right) \tag{2.3}$$

to vary continuously. The graph of (2.3) before symmetry creation is given in Figure 2 along with the difference between (2.3) before and after symmetry creation. Note how small this difference is.

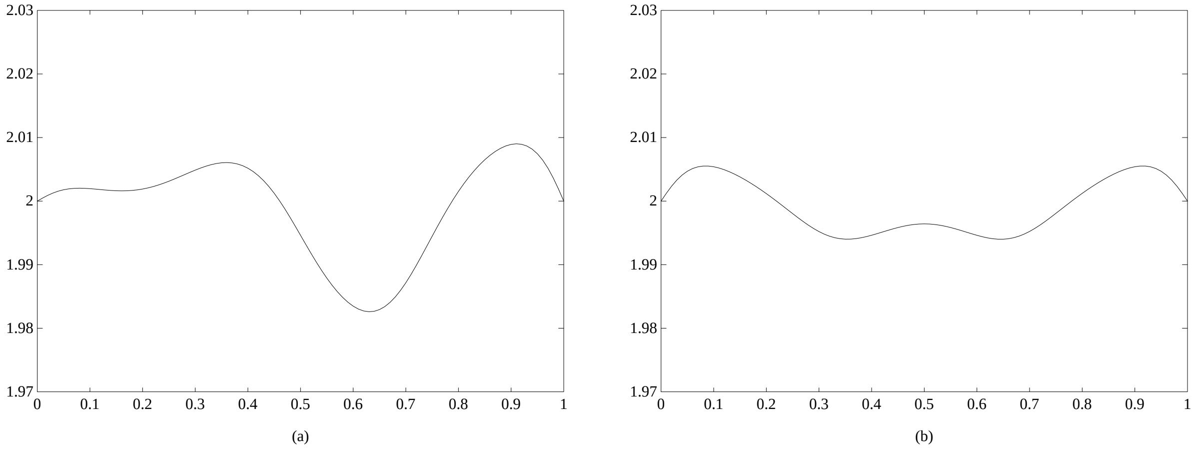

Figure 1: Graphs of F_u for (a) $\lambda = 1.45$, (b) $\lambda = 1.47$.

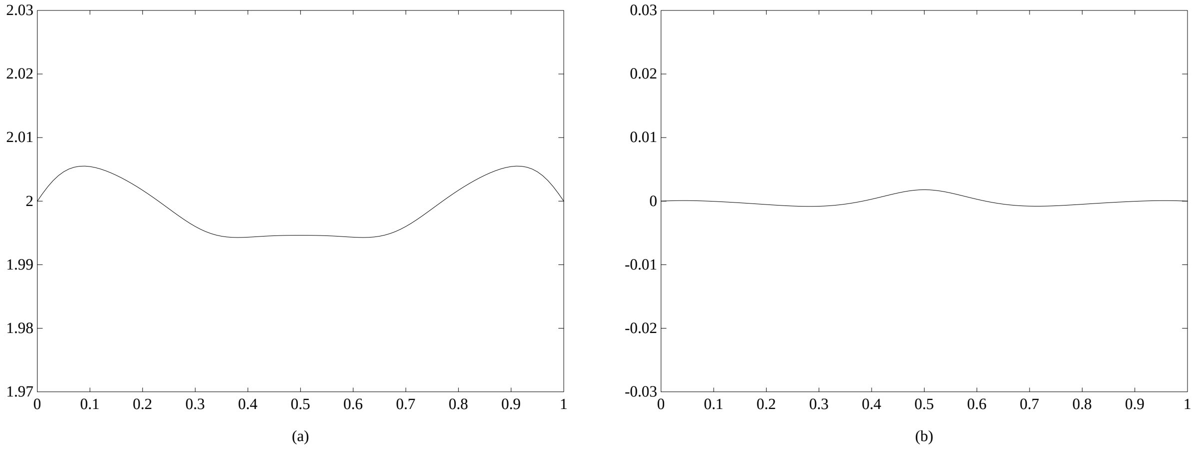

Figure 2: (a) The average of the graphs of F_u for the two conjugate attractors for $\lambda = 1.45$, and (b) the difference between the graph of F_u for $\lambda = 1.47$ and this graph.

(b) The Ginzburg-Landau equation

As a second example of symmetry creation in PDEs we consider the *Ginzburg-Landau equation*:

$$\frac{\partial A}{\partial t} = q^2(i + c_0)\frac{\partial^2 A}{\partial x^2} + \rho A + (i - \rho)A|A|^2 \,. \tag{2.4}$$

The constants q, c_0, ρ are real whereas A is complex. We restrict our attention to Dirichlet boundary conditions:

$$A(0,t) = A(\pi,t) = 0\,.$$

We regard q as the bifurcation parameter and choose the remaining constants as in [7], namely

$$c_0 = 0.25, \quad \rho = 0.25.$$

In this example we compute the averaged quantity

$$F_A(x) = \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T |A(x,\tau)|^2 d\tau\right) \tag{2.5}$$

for the detection of symmetry increasing bifurcations. The results are shown in Figures 3 and 4. Again, $T = 20000$ was chosen for the numerical computations. In contrast to the symmetry creation in the Brusselator the results in this case indicate that there is no continuous transition from the averaged graphs before the bifurcation to the symmetric graph of F_A afterwards (see Figure 4). Similar phenomena corresponding to the *explosion of attractors* have been observed in discrete dynamical systems (see Section 3).

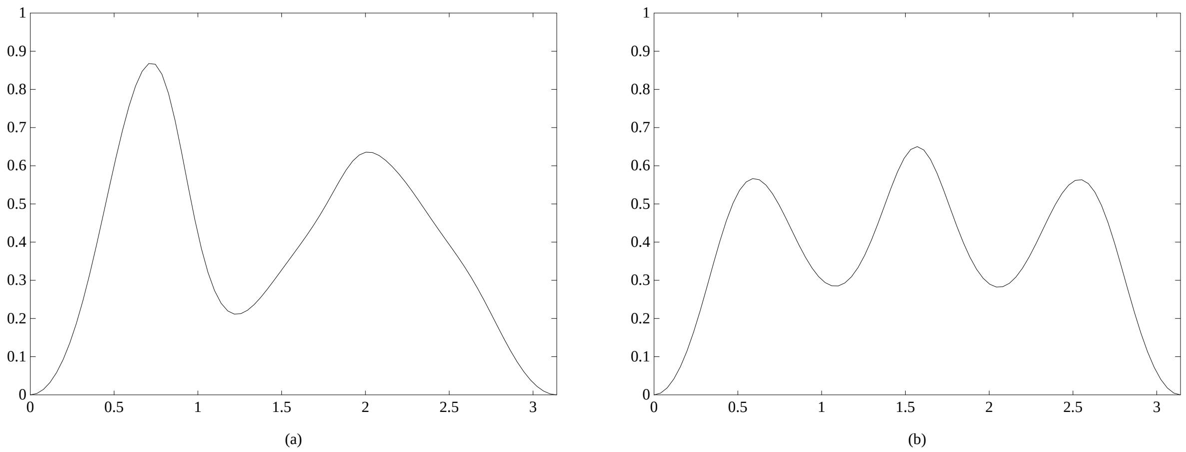

Figure 3: Graphs of F_A for (a) $q = 0.1975$, (b) $q = 0.1925$.

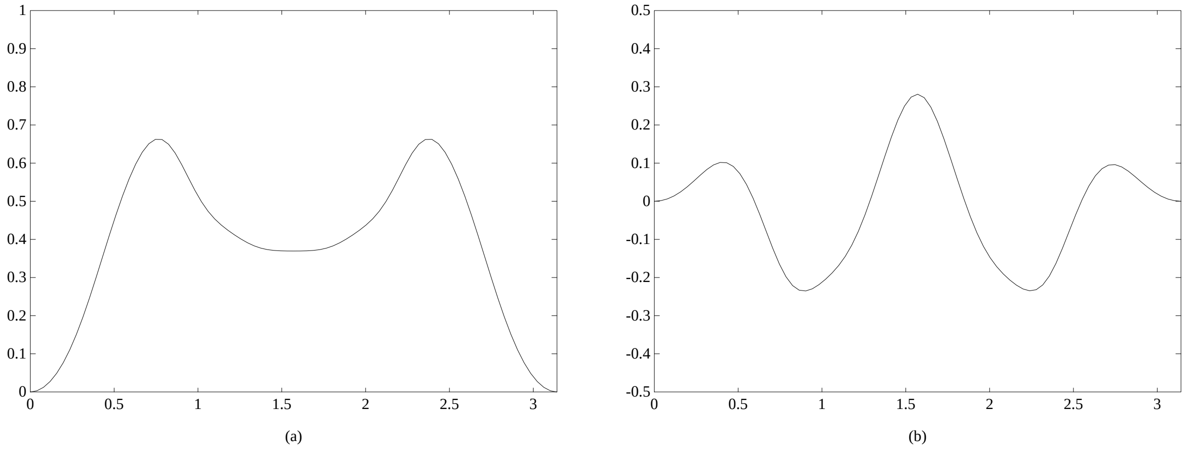

Figure 4: (a) The average of the graphs of F_A for the two conjugate attractors for $q = 0.1975$, and (b) the difference between the graph of F_A for $q = 0.1925$ and this graph.

3 Symmetry creation in mappings

In this section, we consider symmetry increasing of attractors for C^1 mappings $f : \mathbb{R}^m \to \mathbb{R}^m$. The technique that we shall describe is documented in [2] and allows us in certain examples to compute the point at which symmetry increasing occurs to any required accuracy. Our aim here is to illustrate by example both the utility and the limitations of the technique.

Given a closed set $S \subset \mathbb{R}^m$, define $\mathcal{P}_S$ to be the set of all points in $\mathbb{R}^m$ that either lie in S or eventually iterate under f to a point in S. The idea is that S can be chosen so that there is a relation between symmetry increasing of attractors and transitions of the set $\mathcal{P}_S$.

Consider first the case of mappings of the line. Suppose that $f_\lambda : \mathbb{R} \to \mathbb{R}$ is a parametrized family of $\mathbb{Z}_2$-equivariant (odd) mappings. In this case S is chosen to be the origin or, more generally, a symmetric periodic orbit. In many examples, as λ is varied two conjugate asymmetric attractors collide at such an orbit to produce a single $\mathbb{Z}_2$-symmetric attractor containing this orbit.

Let λ_c denote the critical parameter value and suppose that the attractor was asymmetric for $\lambda < \lambda_c$ and $\mathbb{Z}_2$-symmetric for $\lambda > \lambda_c$. In [2] we prove that $A \cap \overline{\mathcal{P}_S} = \emptyset$ for $\lambda < \lambda_c$ and $A \subset \overline{\mathcal{P}_S}$ for $\lambda > \lambda_c$. Thus we search for transitions in the set $\mathcal{P}_S$. Determining the value of λ where such a transition occurs is a standard problem in numerical bifurcation theory. A point x is called a *transition point* at λ_c if there is a positive integer m such that

$$f_{\lambda_c}^m(x) \quad \in \quad S \tag{3.6}$$

$$\frac{\partial}{\partial x} f^m_{\lambda_c}(x) \quad = \quad 0 \tag{3.7}$$

The point x is a *transition point of order* m if in addition,

$$f^{m-1}_{\lambda_c}(x) \notin S. \tag{3.8}$$

In fact there is a simpler way to find transition points of *minimum* order m_0. That is, at minimum order

$$\frac{\partial}{\partial x} f^{m_0}_{\lambda_c}(x) = 0 \quad \text{iff} \quad \frac{\partial}{\partial x} f_{\lambda_c}(x) = 0.$$

This is easily seen using the chain rule.

As an example, we consider the cubic logistic map $f_\lambda(x) = \lambda x - x^3$ whose bifurcation diagram is shown in Figure 2 of [1]. The lowest order for a transition point is $m = 2$. A simple calculation shows that the only transition points of order 2 are $x = \pm\sqrt{\frac{\sqrt{3}}{2}}$ when $\lambda_c = \frac{3\sqrt{3}}{2}$. It is known that this corresponds to the symmetry increasing bifurcation documented in [1]. This is the first of the symmetry increasing bifurcations that occur in the cubic logistic map, see Figure 5. In this figure the graph of f_λ is shown along with the associated attractor in bold.

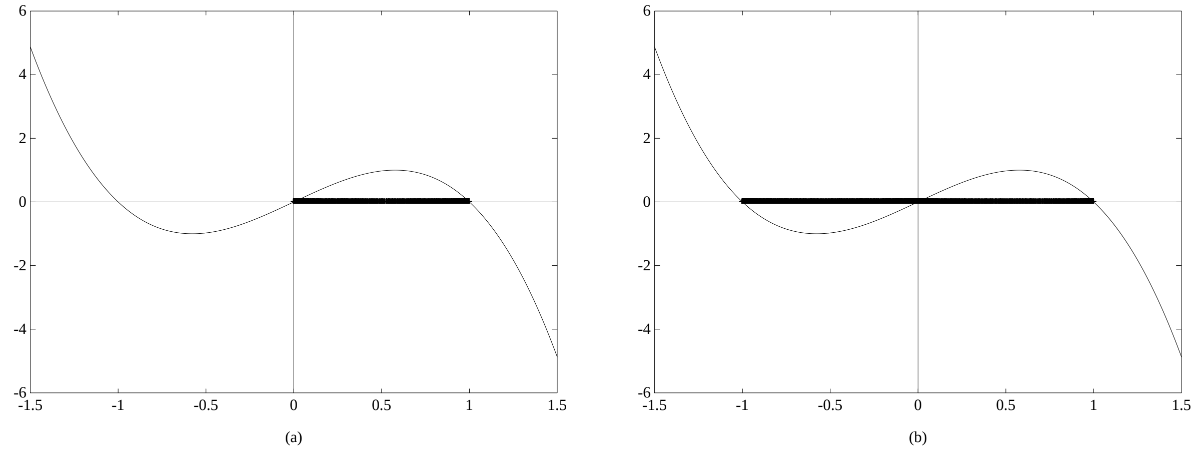

Figure 5: The attractor of the cubic logistic map f_λ for (a) $\lambda = 2.598076$ and (b) $\lambda = 2.598077$.

Further symmetry increasing bifurcations occur after each periodic window in the bifurcation diagram. As an example, the collision of two conjugate attractors at a symmetric period 6 point is shown in Figure 6. Here the corresponding transition point is of order 12. (This can be regarded as a transition point of order 2 for f^6 at a nonsymmetric fixed point.

We end our discussion of one-dimensional mappings by giving an example of a second type of symmetry increasing bifurcation where there is no transition point at criticality.

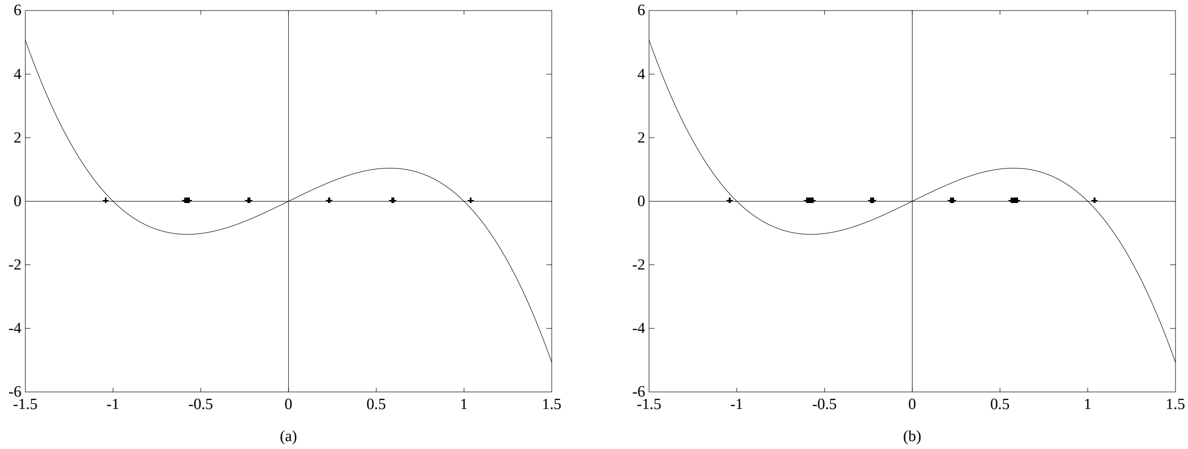

Figure 6: An attractor of the cubic logistic map f_λ for (a) $\lambda = 2.704431$ and (b) $\lambda = 2.704432$.

Rather there is a sequence x_k of transition points of order m_k at λ_k where $m_k \to \infty$ and $\lambda_k \to \lambda_c$. This symmetry increasing bifurcation was observed in the family f_λ^2, where f_λ is the cubic logistic map. It differs from the others in that the asymmetric attractors do not continuously approach a symmetric periodic point for $\lambda < \lambda_c$. Rather they explode to a $\mathbb{Z}_2$-symmetric attractor containing 0 when $\lambda \geq \lambda_c$ (see Figure 7). It should be pointed out that this behavior is not related to a hysteresis (observe that the nonsymmetric attractor before bifurcation is covered by the symmetric one after symmetry creation).

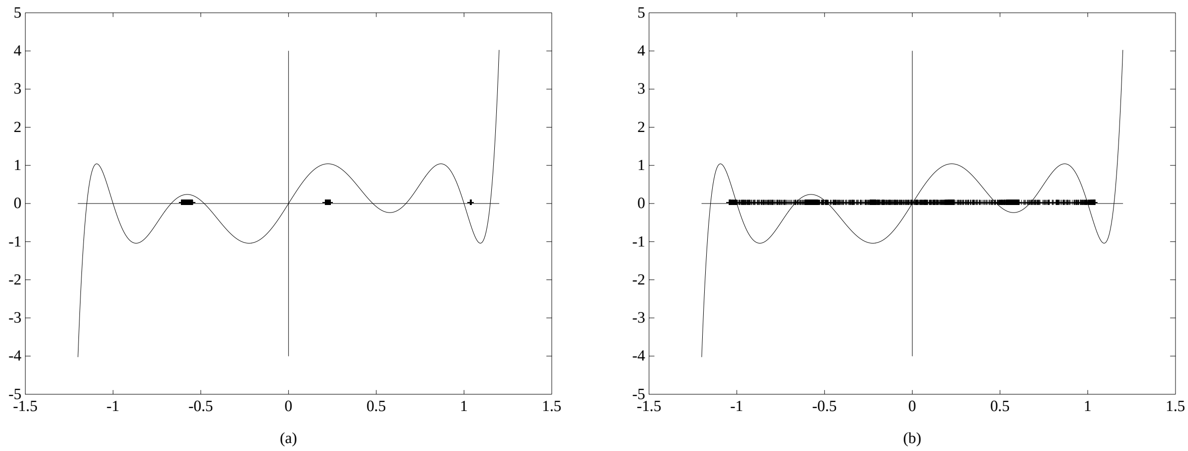

Figure 7: An attractor of f_λ^2 for (a) $\lambda = 2.705639$ and (b) $\lambda = 2.705640$.

Next we consider $\mathbf{D}_3$, the symmetry group of a regular triangle, acting on $\mathbb{R}^2$. The results in [2] suggest that a symmetry increasing bifurcation to an attractor with full $\mathbf{D}_3$-symmetry should be accompanied by a transition in a suitable preimage set $\mathcal{P}_S$, i.e. $A \cap \overline{\mathcal{P}_S} = \emptyset$ before and $A \subset \overline{\mathcal{P}_S}$ after symmetry creation. Here, S is the union of any two axes of symmetry not intersecting the attractor before symmetry creation (two such axis exist by a result in [1]).

As an example we consider the $\mathbf{D}_3$-equivariant map (see [1])

$$f(z, \lambda) = (\alpha u + \beta v + \lambda) z + \gamma \bar{z}^2 \ , \tag{3.9}$$

where $z \in \mathbb{C}$, $\lambda \in \mathbb{R}$ is the bifurcation parameter, $\alpha, \beta, \gamma \in \mathbb{R}$ are fixed constants and

$$u = z\bar{z} \ , \qquad v = \frac{(z^3 + \bar{z}^3)}{2} \ .$$

The group $\mathbf{D}_3$ acts by

$$\kappa z = \bar{z}, \quad \theta z = e^{i\theta} z,$$

where $\theta = \frac{2\pi}{3}$. We use the system

$$\begin{aligned} \mathrm{Im}(f^m)(z, \lambda) &= 0 \\ \frac{\partial}{\partial z}\mathrm{Im}(f^m)(z, \lambda) &= 0 \end{aligned} \tag{3.10}$$

in order to compute symmetry increasing bifurcations. Some numerical results are given in Table 1. In analogy to (3.8) "order" means the smallest value of m for which a solution of (3.10) can be found numerically and which corresponds to a symmetry increasing bifurcation. Infinite order in the last line of the table corresponds to a symmetry creation caused by explosion. For all the bifurcations the attractor has $\mathbb{Z}_2$-symmetry before the symmetry increasing bifurcation and $\mathbf{D}_3$-symmetry afterwards.

Order	z	α	β	γ	λ
2	0.516121 + i 0.915104	-1.0	0.0	-0.5	2.269928
2	0.532793 + i 0.751622	1.0	0.0	0.1	-2.371198
3	-0.214232 + i 0.288230	1.8	0.0	1.34164	-1.648899
4	0.753472 + i 0.772307	-1.0	0.1	-0.8	1.519215
11	0.347859 + i 0.917966	-1.1	0.212	0.6	1.891572
24	0.465201 + i 0.159120	1.0	0.0	0.5	-1.798928
∞	—	1.0	0.7	-0.8	$\approx$ -1.98356

Table 1: Data for symmetry-increasing bifurcations

4 Symmetry creation via drifts along group orbits

In the previous sections we have seen two different mechanisms by which symmetry creation can occur in dynamical systems with discrete symmetry: *collisions* of conjugate attractors or *explosions*. In this section we describe another possibility of symmetry creation that can be found in systems with continuous symmetry: the *drifting* of a chaotic attractor along its group orbit.

We consider the following $\mathbf{O}(2)$-equivariant mapping

$$f(z_1, z_2, \lambda) = \begin{pmatrix} (\alpha + \beta_1 u_1 + \gamma_1 v) z_1 + \delta_1 z_2 \\ (\lambda + \beta_2 u_2 + \gamma_2 v) z_2 + \delta_2 z_1 \end{pmatrix},$$

where

$$u_j = |z_j|^2, \quad v = \mathrm{Re}(z_1 \bar{z}_2).$$

Set

$$\alpha = -2.6, \quad \beta_1 = 1.5, \quad \beta_2 = 0.4, \quad \gamma_1 = 0.7, \quad \gamma_2 = 0.5, \quad \delta_1 = -0.5, \quad \delta_2 = 0.3$$

and regard λ as the bifurcation parameter. In Figure 8 the projection of attractors of f onto the z_1-plane is shown for different values of λ. In (a) the attractor is $\mathbb{Z}_2$ symmetric and a small change in λ causes a drift of this attractor along its $\mathbf{SO}(2)$ group orbit. The resulting attractor in (b) then has full $\mathbf{O}(2)$ symmetry.

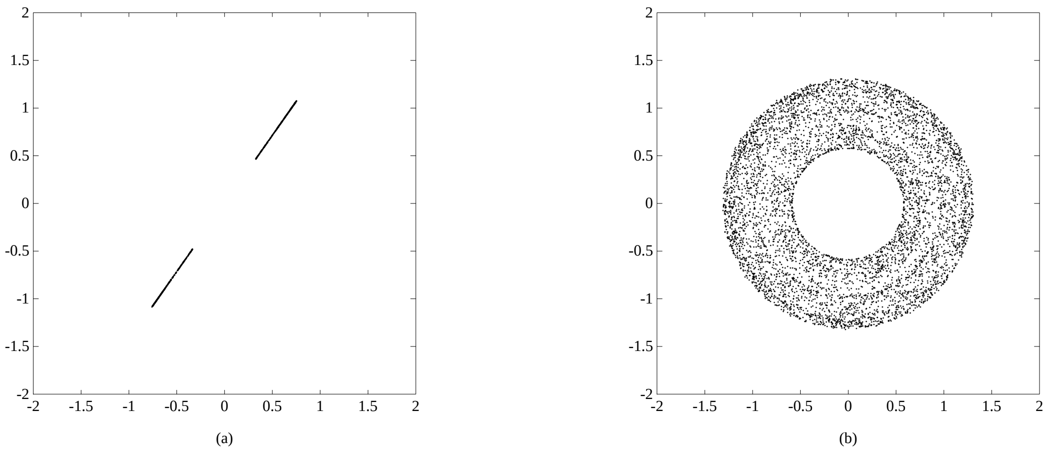

Figure 8: Projection of an attractor of f onto the z_1-plane for (a) $\lambda = -0.477$ and (b) $\lambda = -0.479$.

In [1] the possibility was pointed out that the transition to turbulent Taylor vortices in the Couette-Taylor experiment may be an example of symmetry increasing bifurcation. We will explain why we still believe that this may be true by describing in terms of symmetry the drifting along group orbits that might be responsible for this symmetry creation.

In that experiment a fluid is contained between two concentric circular cylinders with the inner one rotating, at speed (or Reynold's number) λ. When λ is small the flow is laminar Couette flow. As λ is increased Couette flow loses stability to Taylor vortices and then to wavy vortices.

In the analysis of this experiment one often assumes periodic boundary conditions in the axial direction which introduces $\mathbf{O}(2)$ axial symmetry. The total symmetry group is $\mathbf{O}(2) \times \mathbf{SO}(2)$ where the $\mathbf{SO}(2)$ symmetry comes from the azimuthal geometry of the apparatus. In terms of symmetry the solutions described previously have symmetry types

$$\begin{array}{ccccc} \text{Couette flow} & \longrightarrow & \text{Taylor vortices} & \longrightarrow & \text{Wavy vortices} \\ \mathbf{O}(2) \times \mathbf{SO}(2) & \longrightarrow & \mathbb{Z}_2(\kappa) \times \mathbf{SO}(2) & \longrightarrow & \mathbb{Z}_2(\kappa, \pi) \end{array}$$

where κ is a reflection in axial direction and π is a half-period rotation in the azimuthal direction.

As λ is further increased, the flow becomes chaotic and turbulent. However, for large λ, there is a turbulent flow with the pattern of Taylor vortices superimposed. This flow evolves from a turbulent wavy vortex pattern as λ is increased. A transition from turbulent Taylor vortices to homogeneous turbulence takes place at even larger λ.

We believe that the transition from turbulent wavy vortices to turbulent Taylor vortices may be associated with a symmetry increasing bifurcation of a chaotic attractor with $\mathbb{Z}_2(\kappa, \pi)$ symmetry forming a chaotic attractor with $\mathbb{Z}_2(\kappa) \times \mathbf{SO}(2)$ symmetry. Such a change could in principle be generated by drifting along the azimuthal $\mathbf{SO}(2)$ group orbits. Much investigation is needed in order to verify such a mechanism.

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