The Calculus of Residues

Every elementary text in mathematical physics has a section on the calculus of residues because it is a way of finding formulae for integrals of analytic functions that cannot be evaluated otherwise (except maybe numerically).

Moreover it leads to formulae for the number of solutions of an equation $f(z) = 0$ inside a simple closed loop γ by just evaluating an integral whose value must be an integer. You can google, or search Wikipedia, for **Residue theorem, argument principle, Rouché's theorem and winding number** for more information.

An application is the Nyquist stability criterion which is fundamental in electrical engineering.

The **Residue theorem** applies if you want to evaluate an integral of the form $\int_\Gamma\ f(z)\,dz$ where (*) f(z) is analytic on, and inside, Γ except for a finite number of isolated singularit1es $z_1,\ldots,z_J$.

For such functions f there is a Laurent expansion for f(z) that converges on a deleted disk centered at each z_j. Let C be a positively oriented circle center z_0 that is strictly inside Γ, then

$$f(z) \;=\; \sum_{j=1}^{\infty}\ \frac{a_{-j}}{(z-z_0)^j} \;+\; \sum_{j=0}^{\infty}\ a_j(z-z_0)^j$$

for all z inside C. Now $\int_\Gamma\ f(z)\,dz \;=\; \int_C\ f(z)\,dz$ from Cauchy's integral theorem (deformation invariance). This integral around C can be evaluated from the formulae for $\int_C\ (z-z_0)^j\,dz$, so

$$\int_\Gamma\ f(z)\,dz \;=\; 2\pi\,i\,a_{-1}$$

The complex number a_{-1} in the Laurent expansion of $f(z)$ about z_0 is called the **residue of f at** z_0. It is denoted **Res(f;** z_0**)** .

When $f(.)$ has a simple pole at z_0, then the residue is

$$Res(f; z_0) \;=\; \lim_{z \to z_0} (z - z_0)\, f(z)$$

In particular if $f(z) = p(z)/q(z)$ and z_0 is a simple zero of $q(z)$, then

$$Res(f; z_0) \;=\; \frac{p(z_0)}{q'(z_0)}$$

When $f(.)$ has a pole of order m at z_0, then the residue is

$$Res(f : z_0) \;=\; \lim_{z \to z_0} \frac{1}{(z - z_0)^m} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m\, f(z)$$

Cauchy's Residue Theorem

Suppose Γ is a simple, positively oriented closed contour (spocc) and $f(z)$ is analytic inside and on Γ. except at a finite number of singularities at $z_1, \ldots, z_J$ inside Γ. Then

$$\int_\Gamma f(z)\, dz \;=\; 2\pi i \sum_{j=1}^{J} Res(f; z_j)$$

There is a lot available on the internet for example at blog.wolframalpha.com and complex analysis.

Evaluation of some Trigonometric Integrals

Consider the problem of evaluating

$$\int_0^{2\pi} v(\cos\theta, \sin\theta)\ d\theta$$

where v is an analytic function. The functions $\cos\theta, \sin\theta$ are the values of the functions

$$f_1(z) := \frac{1}{2}(z + \frac{1}{z}) \quad \text{and} \quad f_2(z) := \frac{1}{2i}(z - \frac{1}{z})$$

on the unit circle C. ... (Write down a formula for points in C and substitute.) Define

$$F(z) \ := \ \frac{1}{iz} v\left(\frac{1}{2}(z + \frac{1}{z}), \frac{1}{2i}(z - \frac{1}{z})\right)$$

Then the original integral is given by the contour integral of F around C

$$\int_0^{2\pi} v(\cos\theta, \sin\theta)\ d\theta = \int_C F(z)dz.$$

That is it can be expressed as a contour integral which can be evaluated by the calculus of residues. To evaluate these integrals,

(i) determine $F(z)$ from the formula above.

(ii) Find the poles of F inside C and evaluate the residues at these poles,

(iii) Use the residue theorem to evaluate the integral.

Sections 6.3 and 6.4 of Kwok are about using the residue theorem to evaluate certain types of integrals by using complex variable methods and contour integrals. See also the homework examples.

Suppose $f(z)$ is a function that is analytic on a domain D except possibly for isolated poles in D. Such a function is called **meromorphic on D**

When $f(z)$ has a zero of order m at a point z_0, then $f(z) = (z - z_0)^m g(z)$ with g analytic near z_0 and $g(z_0) \neq 0$. In this case z_0 is said to be a **zero of multiplicity m.**

When $f(z)$ has a pole of order m at a point z_0, then

$$f(z) = \frac{g(z)}{(z - z_0)^m} \quad \text{with} \quad g(z_0) \neq 0$$

and g analytic near z_0. Such a z_0 is said to be a **pole of multiplicity m.**

The number of zeroes (poles) of a meromorphic function $f(z)$ inside a closed loop C is

$$N_z(f) := \sum_{z_j \ is \ a \ zero} m_j, \qquad N_p(f) := \sum_{z_j \ is \ a \ pole} m_j$$

where m_j is the multiplicity of z_j as a zero, (pole) of f.

Theorem: If $f(z)$ is meromorphic inside a spocc C and analytic and nonzero on C, then

$$N_z(f) \ - \ N_p(f) \ = \ \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \, dz$$

Corollary: If $f(z)$ is analytic inside and on a spocc C and nonzero on C, then

$$N_z(f) \ = \ \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \, dz$$

Both $N_z(f), N_p(z)$ are positive integers so these integrals must be whole numbers. Numerically one does not have to compute them very accurately. If one uses approximate numerical integration and gets an answer of 3.2 ± 0.4, then the actual value has to be 3. It used to be that computing an integral was easier than solving equations but the following result has always been used extensively.

Theorem (Rouché) Suppose that f, h are analytic functions inside and on a spooc C. If $|h(z)| < |f(z)|$ on C then $f, f+h$ have the same number of zeros inside C.

Example. Show that there are 4 solutions of the equation
$6z^4 + z^3 - 2z^2 + z = 1$ inside the unit disk B_1.

Take this function to be the $f(z) + h(z)$, and choose a simple function $f(z)$ such that the theorem can be used. Then show that the difference $h(z)$ obeys the conditions of the theorem.

When Γ is a closed curve in the complex plane and $z_0 \notin \Gamma$ then the **winding number (or index) of Γ about** z_0 is the value of the integral

$$Ind_\Gamma(z_0) \;:=\; \frac{1}{2\pi i} \int_\Gamma \frac{dz}{z - z_0}$$

This integral is always an integer (possibly negative).

It counts the number of times a closed curve "winds around" a point z_0. it is negative if the curve goes clckwise. If a closed contour is not simple then the winding number must be included in contour integral formulae.