Section 4.4
Rational Functions and Their Graphs

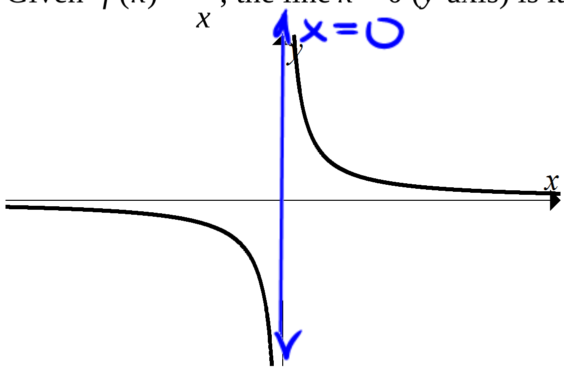
A **rational function** can be expressed as $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not equal to 0. For example, $f(x) = \frac{x-3}{x^2-16}$ is a rational function.

Vertical Asymptote of Rational Functions

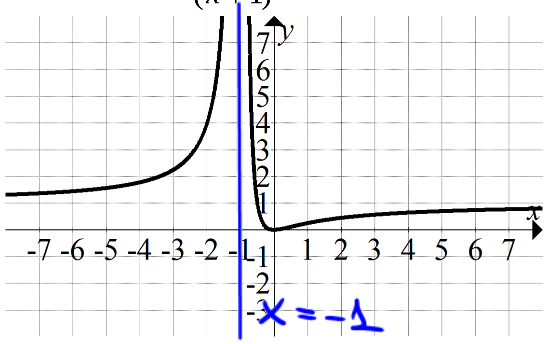
The line $x = a$ is a **vertical asymptote** of the graph of a function f if $f(x)$ increases or decreases without bound as x approaches a.

Examples:

Given $f(x) = \frac{1}{x}$, the line $x = 0$ (y-axis) is its vertical asymptote.



Given $f(x) = \frac{x^2}{(x+1)^2}$, the line $x = -1$ is its vertical asymptote.



Finding Vertical Asymptotes and Holes Algebraically

1. Factor the numerator and denominator as much as possible.
2. Look at each factor in the denominator.
   - If a factor cancels with a factor in the numerator, then there is a hole where that factor equals zero.
   - If a factor does not cancel, then there is a vertical asymptote where that factor equals zero.

Example 1: Find any vertical asymptotes and/or holes of:

$$f(x)=\frac{x^2-3x-10}{x^2-x-6} = \frac{(x-5)\cancel{(x+2)}}{(x-3)\cancel{(x+2)}} = \frac{x-5}{x-3}$$

Vertical Asymptotes:

$x-3=0$

$\boxed{x=3}$

Holes:

$x+2=0$

$x=-2$

$y=\frac{-2-5}{-2-3}=\frac{-7}{-5}=7/5$

$\boxed{(-2, 7/5)}$

Example 2: Find any vertical asymptotes and/or holes of:

$$f(x)=\frac{2x^2+5x-3}{x^3-3x^2-4x} = \frac{2x^2+5x-3}{x(x^2-3x-4)} = \frac{(2x-1)(x+3)}{x(x-4)(x+1)}$$

Vertical Asymptotes:

$\boxed{x=0}$ $x-4=0$ $\boxed{x=4}$ $x+1=0$ $\boxed{x=-1}$

$2x^2+5x-3$

$2(-3)$

$-6 \; 6$

$5 \; -1$

$\underbrace{2x^2+6x}\underbrace{-x-3}$

$\underbrace{2x(x+3)}\underbrace{-1(x+3)}$

$(x+3)(2x-1)$

Holes:

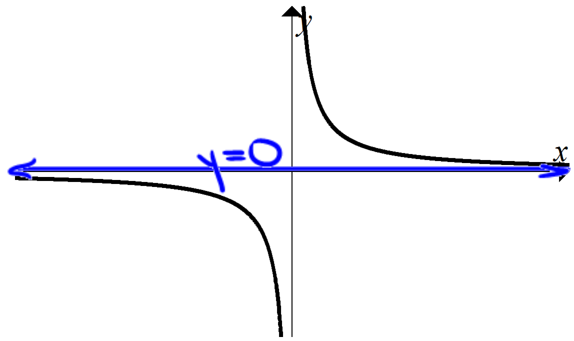
NONE

Horizontal Asymptote of Rational Functions

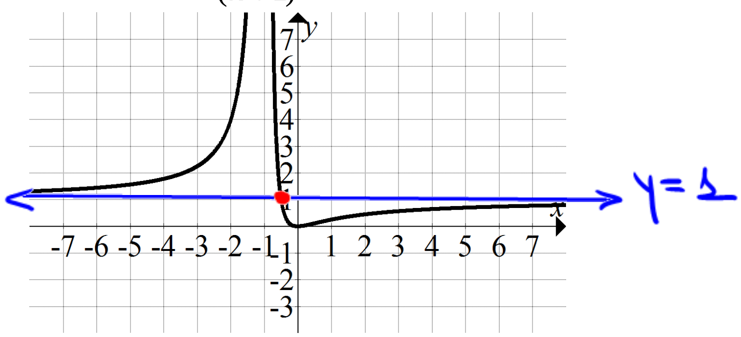
The line $y = b$ is a **horizontal asymptote** of the graph of a function f if $f(x)$ approaches b as x increases or decreases without bound.

Examples:

Given $f(x) = \frac{1}{x}$, the line $y = 0$ (x-axis) is its horizontal asymptote.



Given $f(x) = \frac{x^2}{(x+1)^2}$, the line $y = 1$ is its horizontal asymptote.



Horizontal asymptotes really have to do with what happens to the y-values as x becomes very large or very small. If the y-values approach a particular number at the far left and far right ends of the graph, then the function has a horizontal asymptote.

0 or 1

Note: A rational function may have several vertical asymptotes, but only at most one horizontal asymptote. Also, a graph cannot cross a vertical asymptote, but may cross a horizontal asymptote.

Finding Horizontal Asymptotes

Let $f(x) = \dfrac{p(x)}{q(x)}$. Shorthand: degree of f = deg(f), numerator = N, denominator = D

1. If deg(N) > deg(D) then there is no horizontal asymptote.
2. If deg(N) < deg(D) then there is a horizontal asymptote and it is $y = 0$ (x-axis).
3. If deg(N) = deg(D) then there is a horizontal asymptote and it is $y = \dfrac{a}{b}$, where

   a is the leading coefficient of the numerator.
   b is the leading coefficient of the denominator.

Example 3: Find the horizontal asymptote, if there is one, of:

a. $f(x) = \dfrac{x^9 + 3x^2}{x^6 + 4x^5 - 5}$ deg(N) $>$ deg(D); therefore, no asymptote

9 6

b. $f(x) = \dfrac{x^2 + 2x + 3}{2x^2 + 6x - 1}$ deg(N) $=$ deg(D); therefore, $y = \frac{1}{2}$

2 2

c. $f(x) = \dfrac{1 \cdot x^0}{-x^3 + 5x}$ deg(N) $<$ deg(D); therefore, $y = 0$

0 3

For those examples above that do have a horizontal asymptote, determine whether or not the graph of the function crosses it.

set f(x) = horizontal asymptote.
→ If there is a solution, then crosses.
→ No solution, then doesn't cross

c) $\dfrac{1}{-x^3+5x} = 0$

$1 \neq 0$

No solution, hence no intersection

b) $\dfrac{x^2+2x+3}{2x^2+6x-1} \cancel{=} \dfrac{1}{2}$

$2(x^2+2x+3) = 2x^2+6x-1$

$\cancel{2x^2}+4x+6 = \cancel{2x^2}+6x-1$

$4x+6 = 6x-1$

$6 = 2x-1$

$7 = 2x$

$\boxed{x = \frac{7}{2}}$

Steps to Graphing a Rational Function

1. Factor the numerator and denominator as much as possible. Look at the denominator.
   - If a factor in the numerator ***cancels*** with a factor in the denominator, then there is a **hole** in the graph when that cancelled factor is equal to zero.
   - If a factor ***does not cancel***, then there is a **vertical asymptote** where that factor is equal to zero.

2. Find x-intercept(s) by setting numerator equal to zero.

3. Find the y-intercept (if there is one) by substituting $x = 0$ in the function.

4. Find the horizontal asymptote (if there is one).

5. Use the **x-intercept(s) and vertical asymptote(s)** to divide the x-axis into intervals. Choose a test point in each interval to determine if the function is positive or negative there. This will tell you whether the graph approaches the vertical asymptote in an upward or downward direction.

6. Graph! *Except for the breaks at the vertical asymptotes, the graph should be a nice smooth curve with no sharp corners*.

Example 4: Let $f(x) = \dfrac{3x}{x+2}$. Find any holes, vertical asymptotes, x-intercepts, y-intercept, horizontal asymptote, and sketch the graph of the function.

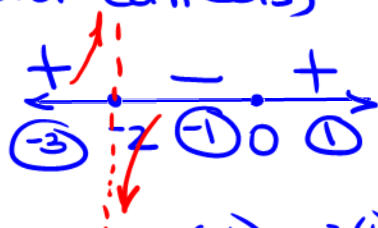
$f(x) = \dfrac{3x}{x+2}$

holes: NONE (NO factor cancels)

vertical asymptotes:

$x+2=0$

$\boxed{x=-2}$



$\frac{3(-3)}{-3+2}$ $\frac{3(-1)}{-1+2}$ $\frac{3(1)}{1+2}$

x-intercepts:

$3x=0$

$x=0$

$\boxed{(0,0)}$

y-intercept:

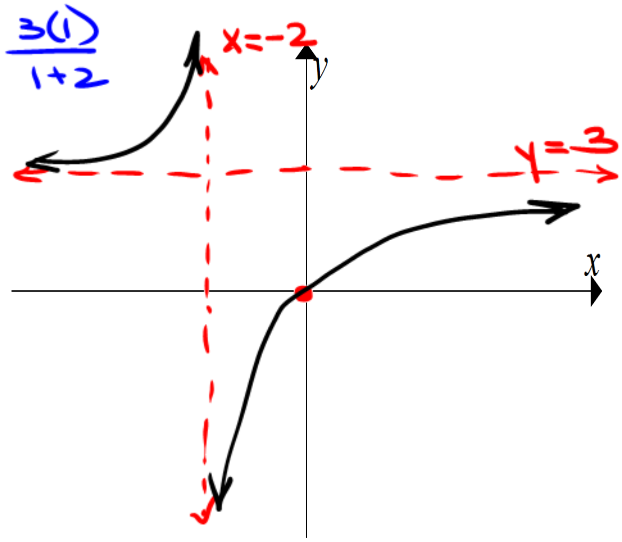
set $x=0$ $\frac{3(0)}{0+2}=0$

$\boxed{(0,0)}$

horizontal asymptote:

deg(N) $\underline{=}$ deg(D); therefore, $\underline{y=3}$

1 1



Example 5: Let $f(x) = \dfrac{x+2}{x-1}$. Find any holes, vertical asymptotes, x-intercepts, y-intercept, horizontal asymptote, and sketch the graph of the function.

$$f(x) = \frac{x+2}{x-1}$$

holes:

NONE

vertical asymptotes:

$x - 1 = 0$

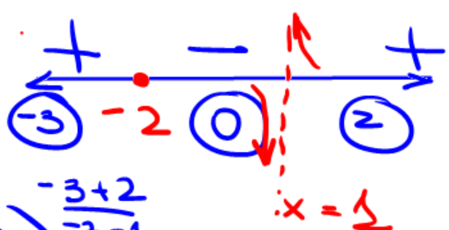
$x = 1$

x-intercepts:

$x + 2 = 0$

$x = -2$

$(-2, 0)$



y-intercept:

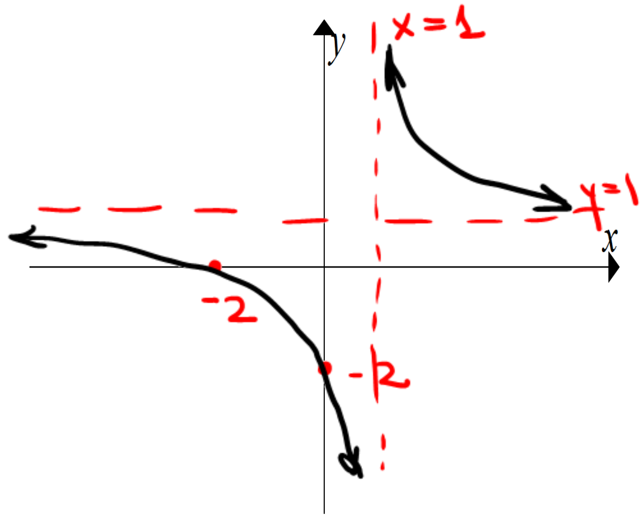
set $x = 0$

$f(0) = \dfrac{0+2}{0-1} = \dfrac{2}{-1} = -2 \quad (0, -2) \quad \dfrac{-3+2}{-3-1}$

horizontal asymptote:

deg(N) $\underline{\;=\;}$ deg(D); therefore, $\underline{\;y = 1\;}$

1 1



Example 6: Let $f(x)=\dfrac{x^2-3x+2}{x^2-4}$. Find any holes, vertical asymptotes, x-intercepts, y-intercept, horizontal asymptote, and sketch the graph of the function.

$$f(x)=\frac{x^2-3x+2}{x^2-4}=\frac{(x-1)\cancel{(x-2)}}{\cancel{(x-2)}(x+2)}=\frac{x-1}{x+2}$$

holes:

$x-2=0$

$x=2$

$y=\dfrac{2-1}{2+2}=\dfrac{1}{4}$

$\left(2,\frac{1}{4}\right)$

vertical asymptotes:

$x+2=0$

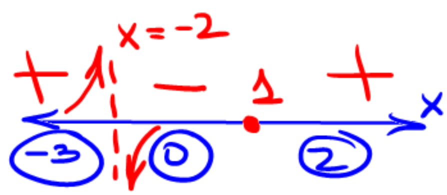
$\boxed{x=-2}$

x-intercepts:

$x-1=0$

$x=1$

$\boxed{(1,0)}$

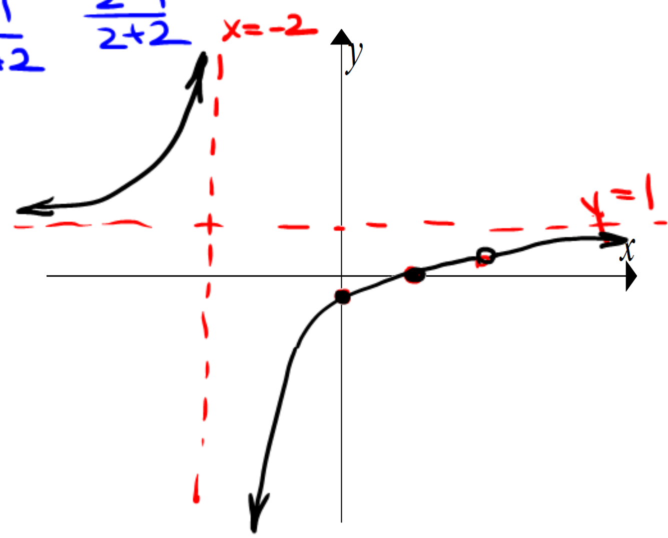


$\dfrac{-3-1}{-3+2}$ $\dfrac{0-1}{0+2}$ $\dfrac{2-1}{2+2}$

y-intercept: set $x=0$

$y=-\frac{1}{2}$

$\boxed{\left(0,-\frac{1}{2}\right)}$



horizontal asymptote:

deg(N) __=__ deg(D); therefore, __$y=1$__

2 2

Try this one: Let $f(x)=\dfrac{-2}{x^2-1}$. Find any holes, vertical asymptotes, x-intercepts, y-intercept, horizontal asymptote, and sketch the graph of the function.

$$f(x)=\frac{-2}{x^2-1}=\frac{-2}{(x-1)(x+1)}$$

holes:

NONE

vertical asymptotes:

$x-1=0$ $\quad$ $x+1=0$

$\boxed{x=1}$ $\quad$ $\boxed{x=-1}$

x-intercepts:

$-2\neq 0$

NONE

y-intercept: set $x=0$

$y=2$

$(0,2)$

$x=-1$ $\quad$ $x=1$

$- \quad + \quad -$

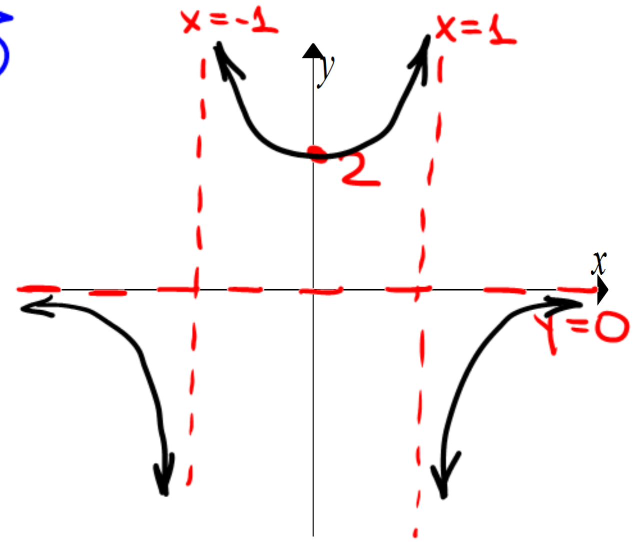
$-2 \quad 0 \quad 2$

$$\frac{-2}{(-2)^2-1} \quad \frac{-2}{(0)^2-1} \quad \frac{-2}{2^2-1}$$

horizontal asymptote:

deg(N) $\underline{\ <\ }$ deg(D); therefore, $\underline{y=0}$

0 $\quad$ 2



Factor

$$P(x) = x^4 + 5x^2 - 50$$

$$x^2 = a$$

$$= (x^2)^2 + 5x^2 - 50$$

$$= a^2 + 5a - 50$$

$$= (a+10)(a-5)$$

$$= (x^2+10)(x^2-5)$$

$$= \left(x^2 + (\sqrt{10})^2\right)\left(x^2 - (\sqrt{5})^2\right)$$

$$= (x + i\sqrt{10})(x - i\sqrt{10})(x - \sqrt{5})(x + \sqrt{5})$$

$x = -i\sqrt{10}$

$x = i\sqrt{10}$

$x = \sqrt{5}$

$x = -\sqrt{5}$

$$P(x) = x^4 - 1$$

$$x^2 = a$$

$$= (x^2)^2 - 1$$

$$= a^2 - 1 = (a-1)(a+1)$$

$$= (x^2-1)(x^2+1)$$

$$= (x-1)(x+1)(x-i)(x+i)$$