Section 4.1
Polynomial Functions and Their Graphs

Definition of a Polynomial Function

Let n be a nonnegative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$, be real numbers, with $a_n \neq 0$. The function defined by $f(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0$ is called a **polynomial function of *x* of degree *n***.

Not allowed

$\frac{1}{x} = x^{-1}$, $\frac{1}{x^2} = x^{-2}$, $\sqrt{x} = x^{1/2}$, $\sqrt[3]{x} = x^{1/3}$

The term $a_n x^n$ is called the **leading term**.

The number a_n, the coefficient of the variable to the highest power, is called the **leading coefficient**.

Polynomials

$x^2 + 3x - 1$

$\sqrt{2}x^5 - .5x + 3$

$x^{10} - 5/7\, x^5 + x$

Not Polynomials

$\sqrt{x} - 3 = x^{1/2} - 3$

$\frac{1}{x} - 5x^2 - 2 = x^{-1} - 5x^2 - 2$

Example 1: Given the following polynomial functions, state the leading term, the degree of the polynomial and the leading coefficient.

a. $P(x) = 6x^4 + 2x^5 + x^2 - 7x + 6$

Leading Term: $2x^5$
Degree: 5
Leading Coefficient: 2

b. $Q(x) = -0.34x - 3x^2 + \sqrt{6} + \frac{1}{2}x^{10}$

Leading Term: $\frac{1}{2}x^{10}$
Degree: 10
Leading Coefficient: $1/2$

c. $g(x) = 2(x+2)(2x-7)^2(x+2)^3$

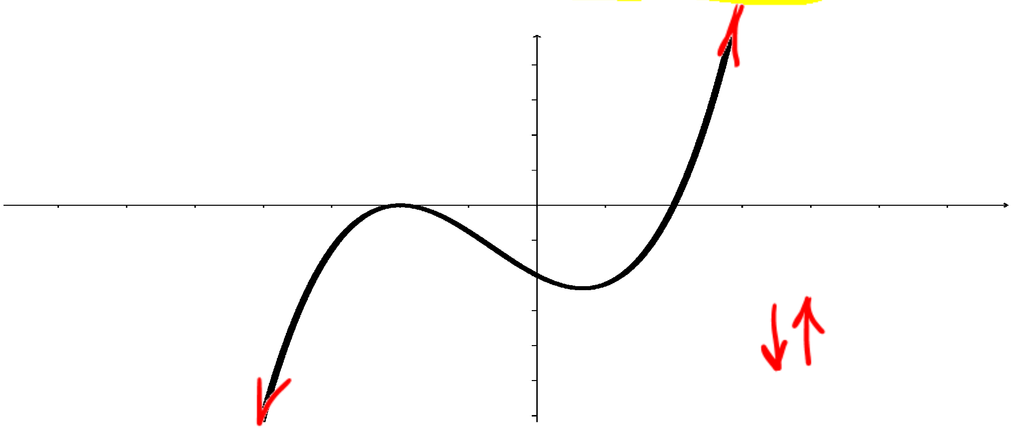
Leading Term: $2x(2x)^2 x^3 = 2x \cdot 4x^2 \cdot x^3 = 8x^6$
Degree: 6
Leading Coefficient: 8

d. $f(x) = (2x-3)(x+9)^4 x(-x-1)^5$

Leading Term: $2x^1 \cdot x^4 \cdot x^1 (-x)^5 = -2x^{11}$
Degree: 11
Leading Coefficient: -2

End Behavior of Polynomial Functions

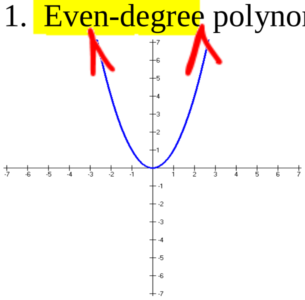
The behavior of a graph of a function to the far left or far right is called its **end behavior.**



$\downarrow\uparrow$

The end behavior of a polynomial function is revealed by the leading term of the polynomial function.

1. Even-degree polynomials look like $y = \pm x^2$ on the ends.



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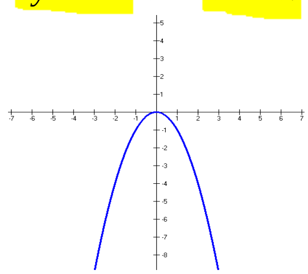

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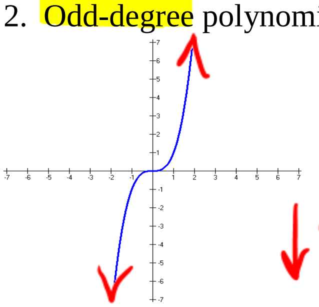
LEADING COEFFIECIENT: + LEADING COEFFICIENT: -

2. Odd-degree polynomials look like $y = \pm x^3$ on the ends.



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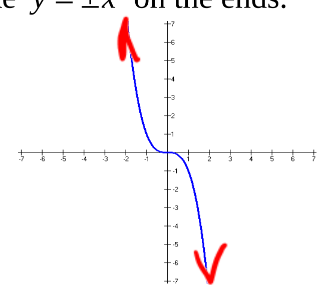

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LEADING COEFFIECIENT: + LEADING COEFFICIENT: -

Example 2: Describe the end behavior of:

a. $P(x) = -x^{11} + 2x$

deg = 11 ← odd

L.C = −1 ← negative $\uparrow\downarrow$

b. $P(x) = (2x-5)^2(x-10)^6$

Leading term $= (2x)^2 \cdot x^6 = 4x^2 \cdot x^6$

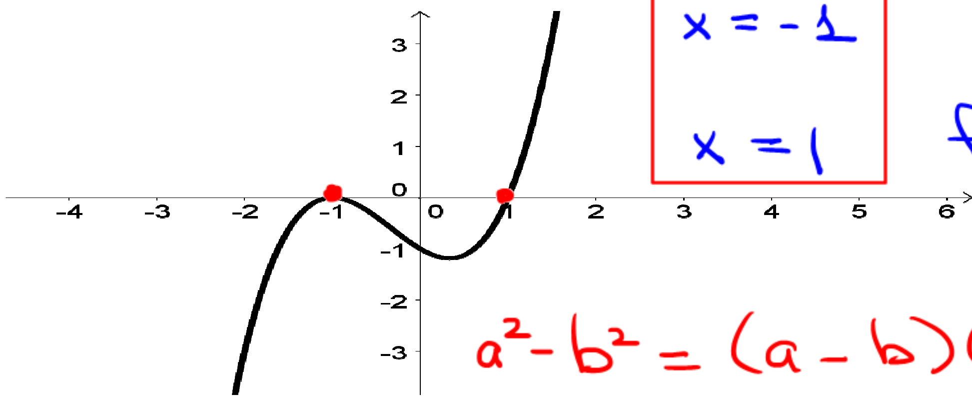
$= 4x^8$

deg = 8 ← even

L.C = 4 ← positive $\uparrow\uparrow$

Zeros of Polynomial Functions

If f is a polynomial function, then the values of x for which $f(x)$ is equal to 0 are called the **zeros** of f.



$\boxed{x = -1 \quad x = 1}$

$f(-1) = 0$

$f(1) = 0$

$a^2 - b^2 = (a-b)(a+b)$

Example 3: Find the zeros of :

a. $f(x) = x^4 - x^2$

$= x^2(x^2 - 1^2)$

$= x^2(x-1)(x+1)$

$x^2 = 0 \qquad x - 1 = 0 \qquad x + 1 = 0$

$\boxed{x = 0 \qquad x = 1 \qquad x = -1}$

b. $f(x) = 3x^2 - 6x - 9$

$= 3(x^2 - 2x - 3)$

$= 3(x-3)(x+1)$

$x - 3 = 0 \qquad x + 1 = 0$

$\boxed{x = 3} \qquad \boxed{x = -1}$

c. $f(x) = -3x\left(x + \frac{1}{2}\right)(x-4)^3$

$-3x = 0 \qquad x + 1/2 = 0 \qquad x - 4 = 0$

$\boxed{x = 0} \qquad \boxed{x = -1/2} \qquad \boxed{x = 4}$

d. $f(x) = 5(4x-2)^4(x+7)^8$

$4x - 2 = 0 \qquad x + 7 = 0$

$+2 \quad +2$

$4x = 2 \qquad \boxed{x = -7}$

$x = \frac{2}{4}$

$\boxed{x = \frac{1}{2}}$

Multiplicity of a Zero

In factoring the equation for the polynomial function *f*, if the same factor $x - r$ occurs *k* times, we call *r* a **repeated zero with multiplicity *k*.**

For example, in the following polynomial function: $f(x) = 3\left(x - \frac{1}{4}\right)^6 (x+10)^5$

the zero $\frac{1}{4}$ has multiplicity 6, and the zero -10 has multiplicity 5.

Example 4: For each of the following polynomials, give the zeros and the multiplicity of each zero.

a. $f(x) = -5(x-2)^5$

$x - 2 = 0$

$\boxed{x = 2 \quad \text{mult.} = 5}$

b. $f(x) = -3x^4(-3x+4)(2x-1)^3$

$x^4 = 0$

$\boxed{x = 0 \quad \text{mult} = 4}$

$-3x + 4 = 0$

$-3x = -4$

$\boxed{x = 4/3 \quad \text{mult} = 1}$

$2x - 1 = 0$

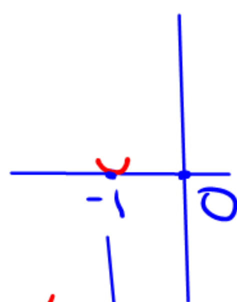
$2x = 1$

$\boxed{x = 1/2 \quad \text{mult} = 3}$

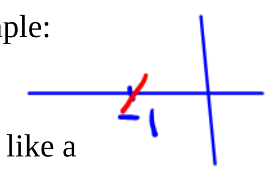
IMPORTANT: The multiplicity of each zero helps us to know what the graph of the function will look like surrounding that zero (*x*-intercept).

Description of the Behavior at Each *x*-intercept

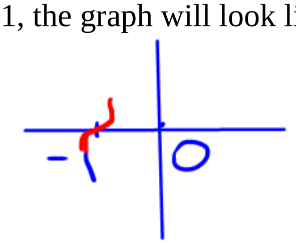
1. Even Multiplicity: The graph touches the *x*-axis, but does not cross it. It looks like a parabola there. Example: For $f(x) = (x+1)^{10}$, around x = -1, the graph will look like:



2. Multiplicity of 1: The graph crosses the *x*-axis. It looks like a line there. Example: For $f(x) = (x+1)$, around x = -1, the graph will look like:



3. Odd Multiplicity greater than or equal 3: The graph crosses the *x*-axis. It looks like a cubic there. Example: For $f(x) = (x+1)^{21}$, around x = -1, the graph will look like:



Steps to Graphing Other Polynomials

1. Determine the **end behavior** by first finding its **leading term**. Is the degree even or odd? Is the sign of the leading coefficient positive or negative? (One of four cases will apply here. See page 2.)

2. Determine the zeros and their multiplicities. If necessary, factor the polynomial.

3. Find the *y*-intercept. set $x=0$

4. Draw the graph, being careful to make a nice smooth curve with no sharp corners.

<u>Note</u>: *without calculus or plotting lots of points, we don't have enough information to know how high or how low the turning points are.*

Example 5: Sketch the graph of $f(x) = -2x^4 + 2x^3$.

Leading Term: $-2x^4$ even $(-)$

End Behavior: (like $-x^2$) $\downarrow \downarrow$

Zeros/Multiplicities:

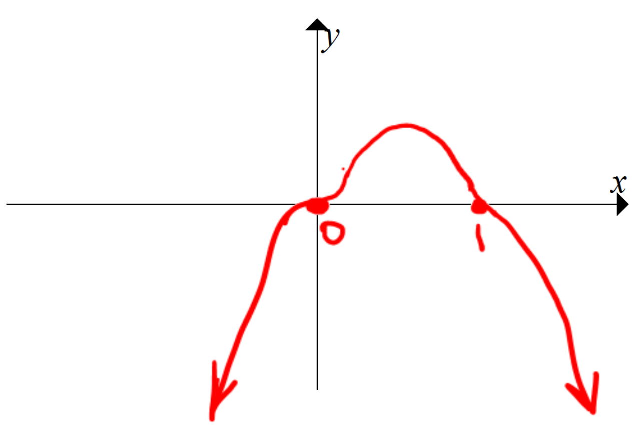
$-2x^4 + 2x^3 = -2x^3(x-1)$

$x = 0$ Mult. = 3 cubic

$x - 1 = 0$, $x = 1$ Mult. = 1 line

y-intercept:

set $x=0$ $f(0) = -2(0)^4 + 2(0)^3 = 0$ $(0,0)$



Example 6: Sketch the graph of $f(x)=(x-2)^3(-x+1)^2(x+4)$.

Leading Term: $x^3(-x)^2\cdot x = x^6$ even (+)

End Behavior: x^2 ↑ ↑

Zeros/Multiplicities:

$x-2=0$
$x=2$
mult. = 3
cubic

$-x+1=0$
$-x=-1$
$x=1$
mult = 2
parabola

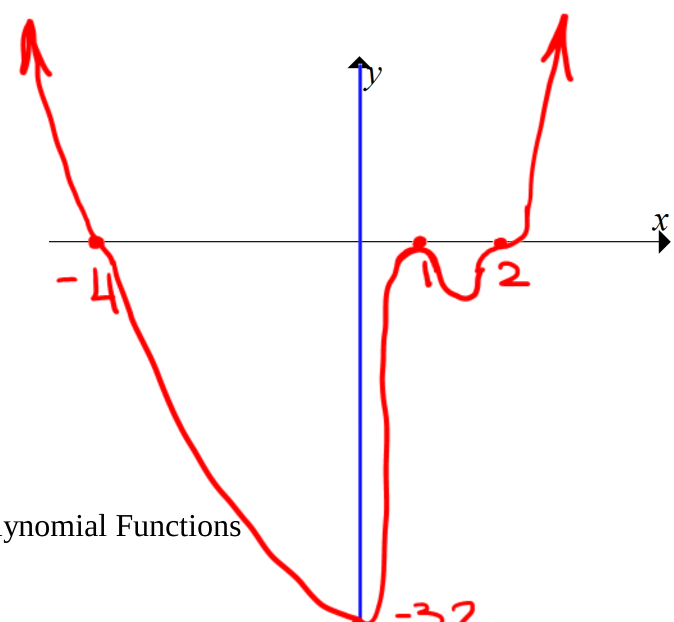
$x+4=0$
$x=-4$
mult. = 1
line

y-intercept:

set $x=0$

$(0-2)^3(0+1)^2(0+4)$
$=(-2)^3(4)=(-8)(4)=-32$

$(0,-32)$



Example 7: Sketch the graph of $f(x) = -x(x+2)^2(x-3)^2$.

Leading Term: End Behavior:

$-x(x)^2(x)^2 = -x^5$ odd $(-)$ $-x^3$ ↑ ↓

Zeros/Multiplicities:

$-x = 0$ $x+2=0$ $x-3=0$

$x = 0$ $x=-2$ $x=3$

mult. = 1 mult. = 2 mult. = 2

line parabola parabola

y-intercept:

set $x=0$ $f(0)=0$ $(0,0)$