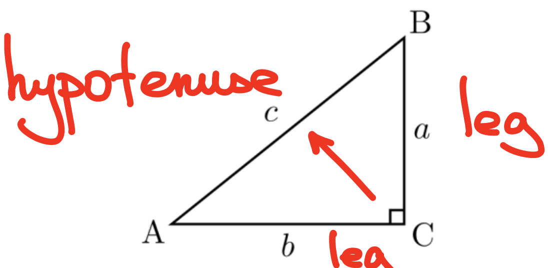
2312 - Section 4.1
Special Triangles and Trigonometric Ratios

First we review some conventions involving triangles and known facts.
It is common to label angles of a triangle with capital letters (Ex.: A, B, and C) and then sides with low case letters a, b, and c with side a opposite angle A, side b opposite angle B, and side c opposite angle C.



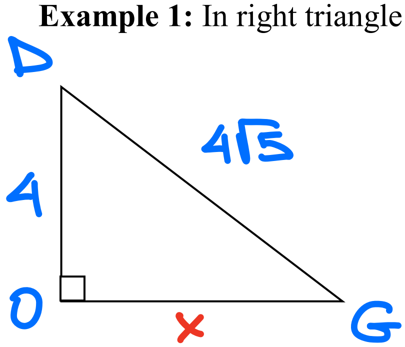
The **sum of measures** of three angles of a triangle is 180°.
An **acute angle** of a triangle is an angle that measures between 0° and 90°.
A **right angle** is a 90° angle.
A right triangle has one right angle. Two other angles are always **acute** and **complementary**, i.e. their measures add up to 90°. The side opposite the right angle is called **hypotenuse**, two other sides are called **legs**.

In this section, we will work with some **special right** triangles before moving on to defining the six trigonometric functions.

If we are given a right triangle and we are interested in finding a missing side, we can apply
Pythagorean's Theorem

$$\boxed{a^2 + b^2 = c^2}$$

Example 1: In right triangle DOG with the right angle O find OG if $DG = 4\sqrt{5}$ and $DO = 4$.



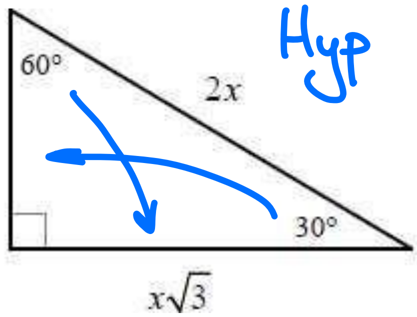
$$4^2 + x^2 = (4\sqrt{5})^2$$
$$16 + x^2 = 80$$
$$\sqrt{x^2} = \sqrt{64}$$
$$x = 8$$
$$\boxed{OG = 8}$$

Two **special** triangles are $30^{\circ} - 60^{\circ} - 90^{\circ}$ triangles and $45^{\circ} - 45^{\circ} - 90^{\circ}$ triangles.
In such triangles, sides are proportional. You need to know the length of one side only to find the remaining sides.

$30^{\circ} - 60^{\circ} - 90^{\circ}$ Triangles

Short leg

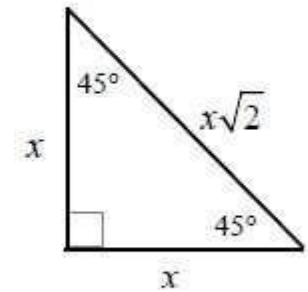
Hyp



Long Leg

$x : x\sqrt{3} : 2x$

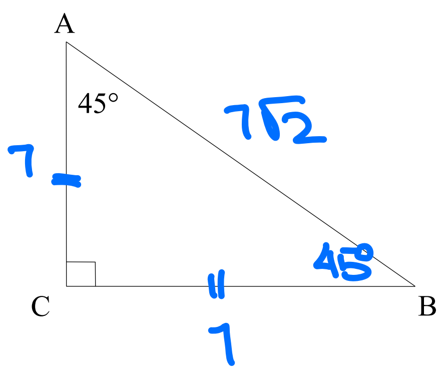
$45^{\circ} - 45^{\circ} - 90^{\circ}$ Triangles



$x : x : x\sqrt{2}$

Example 2:

a) *Given:* Right ΔABC with $\angle C = 90^{\circ}$ and $\angle A = 45^{\circ}$; $AC = 7$. *Find*: AB and BC.



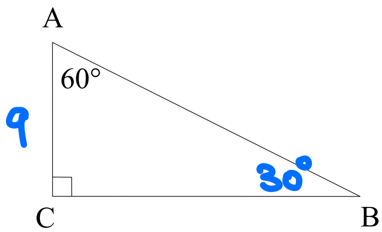
$45° - 45° - 90°$

$x : x : x\sqrt{2}$

$7 : 7 : 7\sqrt{2}$

$$\boxed{\begin{array}{l} AB = 7\sqrt{2} \\ BC = 7 \end{array}}$$

b) *Given:* Right ΔABC with $\angle C = 90^{\circ}$ and $\angle A = 60^{\circ}$; $AC = 9$. *Find*: AB and BC.



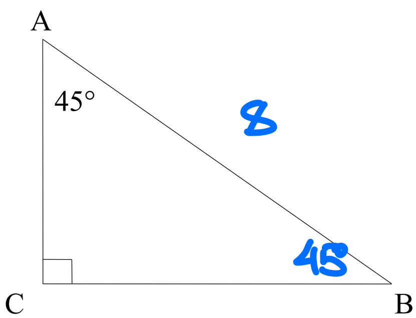
$30^{\circ} - 60^{\circ} - 90^{\circ}$

$x : x\sqrt{3} : 2x$

$9 : 9\sqrt{3} : 18$

$$\boxed{\begin{aligned} AB &= 18 \\ BC &= 9\sqrt{3} \end{aligned}}$$

c) *Given:* Right ΔABC with $\angle C = 90^{\circ}$ and $\angle A = 45^{\circ}$; $AB = 8$. *Find*: AC and BC.



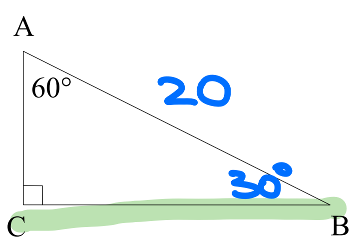
$x : x : x\sqrt{2}$, $x\sqrt{2} = 8$

$AC = BC = 4\sqrt{2}$

$x\sqrt{2} = 8$

$x = \frac{8}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\cancelto{4}{8}\sqrt{2}}{\cancel{2}} = \boxed{4\sqrt{2}}$

d) *Given:* Right ΔABC with $\angle C = 90^{\circ}$ and $\angle A = 60^{\circ}$; $AB = 20$. *Find*: BC



$30^{\circ} - 60^{\circ} - 90^{\circ}$

$x : x\sqrt{3} : 2x$, $2x = 20$

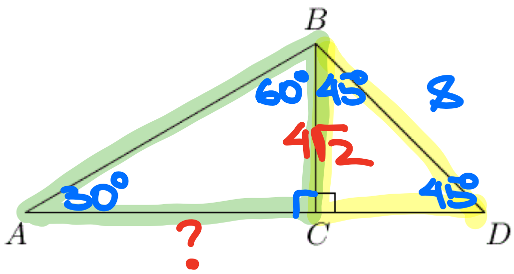
Short Leg First!

$20 = 2x$

$x = 10$

$$\boxed{BC = 10\sqrt{3}}$$

Example 3: Find AC if $\angle A = 30^{\circ}$, $\angle D = 45^{\circ}$, and $BD = 8$.



45-45-90

$x : x : x\sqrt{2}$

$x\sqrt{2} = 8$

$x\sqrt{2} = 8$

$x = \frac{8}{\sqrt{2}} = 4\sqrt{2}$

30-60-90

$x : x\sqrt{3} : 2x$

$x = 4\sqrt{2}$

$\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$

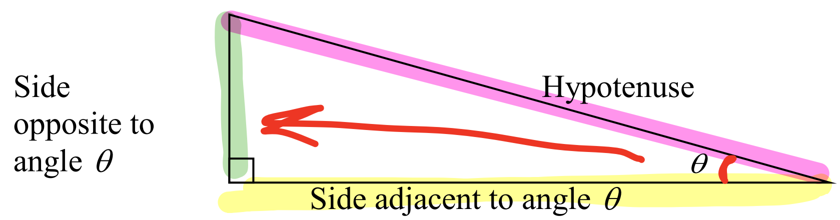
$x = 4\sqrt{2}$

$x\sqrt{3} = 4\sqrt{2} \cdot \sqrt{3} = 4\sqrt{6}$

The Six Trigonometric Functions of an Angle

A trigonometric function is a ratio of the lengths of the sides of a triangle. If we fix an angle, then as to that angle, there are three sides, the **adjacent side**, the **opposite side**, and the **hypotenuse**. We have **six** different combinations of these three sides, so there are six trigonometric functions. The inputs for the trigonometric functions are angles and the outputs are real numbers.

Let θ be an acute angle in a right triangle.



Sine Function:

$$\sin\theta = \frac{\textit{length of the side opposite } \theta}{\textit{length of hypotense}}$$

Cosine Function:

$$\cos\theta = \frac{\textit{length of the side adjacent to } \theta}{\textit{length of hypotense}}$$

Tangent Function:

$$\tan\theta = \frac{\textit{length of the side opposite } \theta}{\textit{length of the side adjacent to } \theta}$$

Cotangent Function:

$$\cot\theta = \frac{\textit{length of the side adjacent to } \theta}{\textit{length of the side opposite } \theta}$$

Secant Function:

$$\sec\theta = \frac{\textit{length of hypotense}}{\textit{length of the side adjacent to } \theta}$$

Cosecant Function:

$$\csc\theta = \frac{\textit{length of hypotense}}{\textit{length of the side opposite } \theta}$$

Note: For acute angles, the values of the trigonometric functions are always positive since they are ratios of lengths.

A useful mnemonic device:

SOH-CAH-TOA

$S = \dfrac{O}{H}$

$C = \dfrac{A}{H}$

$T = \dfrac{O}{A}$

$$\sin\theta = \frac{opposite}{hypotenuse}$$

$$\cos\theta = \frac{adjacent}{hypotenuse}$$

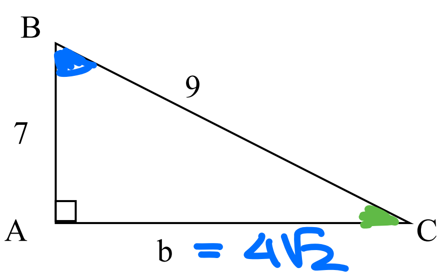
$$\tan\theta = \frac{opposite}{adjacent}$$

$$\cot\theta = \frac{adjacent}{opposite}$$

$$\sec\theta = \frac{hypotense}{adjacent}$$

$$\csc\theta = \frac{hypotense}{opposite}$$

Example 4: Suppose you are given this triangle. Find the six trigonometric functions of angle B and angle C.



$7^2 + b^2 = 9^2 \quad | \quad \sqrt{b^2} = \sqrt{32}$

$49 + b^2 = 81 \quad | \quad b = \sqrt{32} = \sqrt{16\cdot 2}$

$= \sqrt{16}\cdot\sqrt{2} = 4\sqrt{2}$

For $\angle B$

$\sin B = \dfrac{O}{H} = \dfrac{4\sqrt{2}}{9}$

$\cos B = \dfrac{A}{H} = \dfrac{7}{9}$

$\tan B = \dfrac{O}{A} = \dfrac{4\sqrt{2}}{7}$

$\csc B = \dfrac{9}{4\sqrt{2}}\cdot\dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{9\sqrt{2}}{8}$

$\sec B = \dfrac{9}{7}$

$\cot B = \dfrac{7}{4\sqrt{2}}\cdot\dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{7\sqrt{2}}{8}$

For $\angle C$

$\sin C = \dfrac{7}{9}$

$\cos C = \dfrac{4\sqrt{2}}{9}$

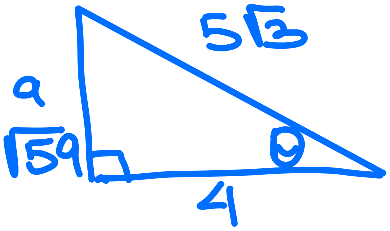
$\tan C = \dfrac{7}{4\sqrt{2}}\cdot\dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{7\sqrt{2}}{8}$

$\csc C = \dfrac{9}{7}$

$\sec C = \dfrac{9}{4\sqrt{2}}\cdot\dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{9\sqrt{2}}{8}$

$\cot C = \dfrac{4\sqrt{2}}{7}$

Example 5: Suppose that θ is an acute angle in a right triangle and $\sec\theta = \frac{5\sqrt{3}}{4}$. Find $\sin\theta$ and $\cot\theta$.



$$\sec\theta = \frac{5\sqrt{3}}{4} = \frac{H}{A}$$

$$a^2 + 4^2 = (5\sqrt{3})^2$$

$$a^2 + 16 = 75$$

$$\sqrt{a^2} = \sqrt{59}$$

$$a = \sqrt{59}$$

$$\sin\theta = \frac{O}{H} = \frac{\sqrt{59}}{5\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{177}}{15}$$

$$\cot\theta = \frac{A}{O} = \frac{4}{\sqrt{59}} = \frac{4\sqrt{59}}{59}$$

Cofunctions
A special relationship exists between sine and cosine, tangent and cotangent, secant and cosecant. The relationship can be summarized as follows:

90 − θ

$$\boxed{\text{Function of } \theta = \text{Cofunction of the complement of } \theta}$$

Recall that two angles are **complementary** if their sum is 90°. In example considered earlier, angles B and C are complementary, since $A = 90^{\circ}$. Notice that $\sin C = \cos B$ and $B = \cos C$. The same will be true of any pair of cofunctions.

Cofunction relationships

$$\begin{aligned}
\sin A &= \cos(90^{\circ} - A) \\
\cos A &= \sin(90^{\circ} - A) \\
\tan A &= \cot(90^{\circ} - A) \\
\cot A &= \tan(90^{\circ} - A) \\
\sec A &= \csc(90^{\circ} - A) \\
\csc A &= \sec(90^{\circ} - A)
\end{aligned}$$

Example 6: Fill in the blanks

a) $\sin 60^{\circ} = \cos$ (90 − 60)

= cos(30°)

b) $\csc 15^{\circ} =$ sec(90 − 15)

sec(75°)