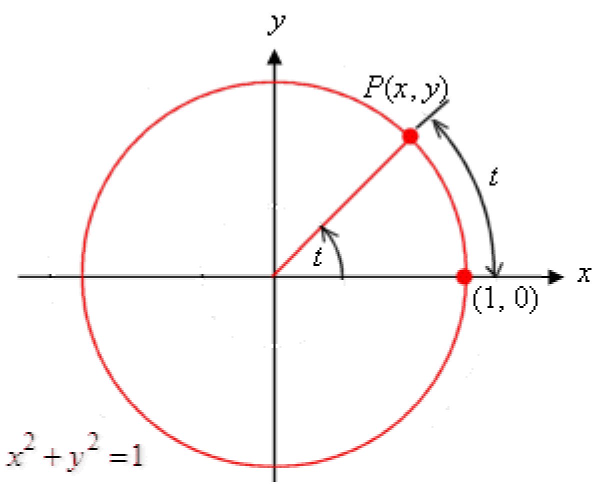
MATH 2312 - Section 5.1 - Trigonometric Functions of Real Numbers

In calculus and in the sciences many of the applications of the trigonometric functions require that the inputs be real numbers, rather than angles. By making this small but crucial change in our viewpoint, we can define the trigonometric functions in such a way that the inputs are real numbers. The definitions of the trig functions, and the identities that we have already met (and will meet later) will remain the same, and will be valid whether the inputs are angles or real numbers.

Let $P(x, y)$ be a point on a unit circle $x^2 + y^2 = 1$ whose arc length from the point $(1,0)$ is t.



The six trigonometric functions of the **real number *t*** are defined as follows.

$\cos(t) = x$	$\sin(t) = y$	$\tan(t) = \frac{y}{x},\ x \neq 0$
$\sec(t) = \frac{1}{x},\ x \neq 0$	$\csc(t) = \frac{1}{y},\ y \neq 0$	$\cot(t) = \frac{x}{y},\ y \neq 0$

Example 1: Given that $\cos(t) = -\frac{1}{4}$ and $\pi < t < \frac{3\pi}{2}$, find $\sin(t)$ and $\tan(t)$.

The identities we discussed earlier work for both angles and real numbers:

Reciprocal Identities

1. $\sec t = \frac{1}{\cos t}, \ \cos t \neq 0$

   $\csc t = \frac{1}{\sin t}, \ \sin t \neq 0$

   $\cot t = \frac{1}{\tan t}, \ \tan t \neq 0$

2. $\frac{\sin t}{\cos t} = \tan t\,; \ \frac{\cos t}{\sin t} = \cot t$

Pythagorean Identities

3. $\sin^2 t + \cos^2 t = 1$

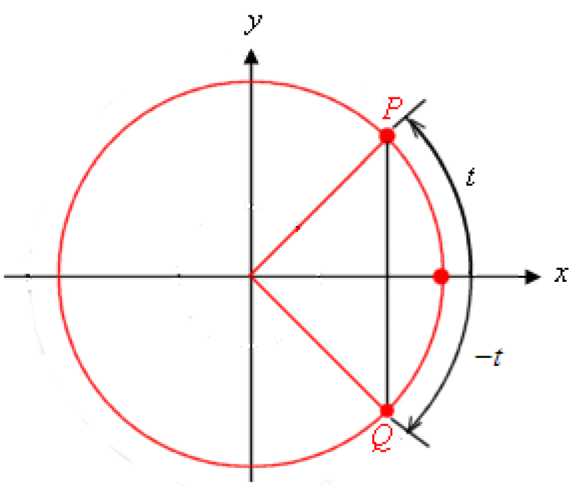
   $\tan^2 t + 1 = \sec^2 t$

   $\cot^2 t + 1 = \csc^2 t$

Opposite Angle Identities

$P: \ (\cos(t), \sin(t))$

$Q: \ (\cos(-t), \sin(-t)).$



$\sin(-x) = -\sin(x)$	$\tan(-x) = -\tan(x)$	$\sec(-x) = \sec(x)$
$\cos(-x) = \cos(x)$	$\cot(-x) = -\cot(x)$	$\csc(-x) = -\csc(x)$

To summarize: Cosine and Secant functions are EVEN functions.
Sine, Tangent, Cotangent, and Cosecant are ODD functions.

Example 2: Use the opposite-angle identities to find/evaluate/simplify.

a. $\sin\left(-\frac{2\pi}{3}\right)$

b. $\cos\left(-\frac{5\pi}{6}\right)$

c. $\tan\left(-\frac{\pi}{4}\right) + \cot\left(-\frac{3\pi}{4}\right)$

Periodicity

If we start with a point P on the unit circle and travel a distance of 2π units, we arrive back at the same point P. Thus, we have the following identities.

$$\cos(t + 2\pi k) = \cos(t) \qquad \sec(t + 2\pi k) = \sec(t)$$
$$\sin(t + 2\pi k) = \sin(t) \qquad \csc(t + 2\pi k) = \csc(t)$$

for all real numbers t and integers k.

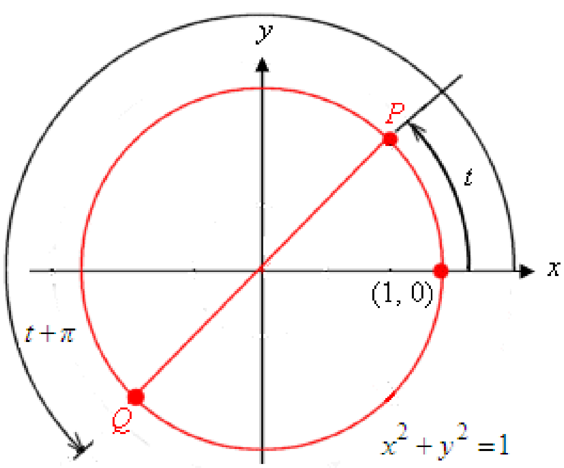
Tangent and cotangent functions also repeat themselves, but at shorter lengths, namely π. Thus, we have the following identities.

$$\tan(t + \pi k) = \tan(t) \qquad \cot(t + \pi k) = \cot(t)$$

for all real numbers t and integers k.

P: $(\cos(t), \sin(t))$

Q: $(\cos(t+\pi), \sin(t+\pi))$.



Example 3: Evaluate the following.

a. $\tan\left(\frac{15\pi}{4}\right)$

b. $\cos\left(\frac{25\pi}{6}\right)$

c. $\sin\left(-\frac{20\pi}{3}\right)$

d. $\cos(11\pi)$

e. $\cos(12\pi)$

f. $\sin(53\pi)$

Note: $\sin(\pi k) = 0$, for ALL integer k.
$\cos(\pi k) = 1$, when k is EVEN and $\cos(\pi k) = -1$, when k is ODD.

Example 4: Evaluate.

$$\frac{\cos\left(\frac{19\pi}{3}\right)\tan\left(\frac{21\pi}{4}\right)}{\cos(8\pi)\sin\left(\frac{25\pi}{6}\right)}$$

Example 5: Simplify the following.

$$\frac{\sin(t+6\pi)\csc(t-2\pi)}{\cot(t+\pi)\tan(t+2\pi)}$$

Example 6: Simplify the following.

$$\cos(-t)+\cos(-t)\tan^2(-t)$$

Example 7: Simplify the following.

$$\frac{\sec(t + 4\pi) + \csc(t + 6\pi)}{1 + \tan(t + 5\pi)}$$

Example 8: Given $\sin(x) = \frac{3}{5}$ and $0 < x < \frac{\pi}{2}$, evaluate:

$$\sin(x + 13\pi) + \cos(x - 14\pi) + \tan(x + 7\pi).$$

Example 9: Given $0 < t < \frac{\pi}{2}$; simplify

$$\frac{\cos\left(\frac{\pi}{2} - t\right)\sqrt{1 + \tan^2(t)}}{\sqrt{9\sec^2(t) - 9}}$$

Recall: **Cofunction relationships**

$\sin x = \cos\left(\frac{\pi}{2} - x\right)$ $\qquad$ $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ $\qquad$ $\tan x = \cot\left(\frac{\pi}{2} - x\right)$