Section 5.2
Graphs of the Sine and Cosine Functions

A Periodic Function and Its Period

A nonconstant function \( f \) is said to be periodic if there is a number \( p > 0 \) such that \( f(x + p) = f(x) \) for all \( x \) in the domain of \( f \). The smallest such number \( p \) is called the period of \( f \).

The graphs of periodic functions display patterns that repeat themselves at regular intervals.

Amplitude

Let \( f \) be a periodic function and let \( m \) and \( M \) denote, respectively, the minimum and maximum values of the function. Then the amplitude of \( f \) is the number \( \frac{M - m}{2} \).

In other words, the amplitude is half the height.

Example 1: Given the following graph, state its amplitude and period.

a.

![Graph 1](image)

\[ A = \frac{2 - (-2)}{2} = 2 \]

\[ p = 2\pi - 0 = 2\pi \]

b.

![Graph 2](image)

\[ A = \frac{8 - (-2)}{2} = 5 \]

\[ p = 4.5 - (-1.5) = 6 \]
Now let’s talk about the graphs of the sine and cosine functions.

We know from previously studying the periodicity of the trigonometric functions that the sine function repeats itself after $2\pi$ radians. Hence, its period is $2\pi$. This also tells us that we can find the whole number line wrapped around the unit circle.

**Sine:** $f(x) = \sin x$

![Sine Graph](image)

Since the period of the sine function is $2\pi$, we will graph the function on the interval $[0, 2\pi]$. The rest of the graph is made up of repetitions of this portion.

![Cosine Graph](image)

Key points in graphing sine functions are obtained by dividing the period into four equal parts. (Assuming no vertical shifting.)

One complete cycle of the sine curve includes three $x$-intercepts, one maximum point, and one minimum point.

The graph has $x$-intercepts at the beginning, middle, and end of its full period.

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We know from previously studying the periodicity of the trigonometric functions that the cosine function repeats itself after $2\pi$ radians. Hence, its **period is $2\pi$**. This also tells us that we can find the whole number line wrapped around the unit circle.

**Cosine:** $f(x) = \cos x$

Since the period of the cosine function is $2\pi$, we will graph the function on the interval $[0, 2\pi]$. The rest of the graph is made up of repetitions of this portion.

Key points in graphing cosine functions are obtained by dividing the period into four equal parts. (Assuming no vertical shifting and no $x$-axis reflection.)

One complete cycle of the cosine curve includes **two $x$-intercepts, two maximum points** and one **minimum point**.

The graph has $x$-intercepts at the second and fourth points of its full period.

**Notice that the graphs of sine and cosine are exactly the same except for a horizontal shift.**
For the following functions:  \( y = A \sin(Bx - C) + D \) and  \( y = A \cos(Bx - C) + D \)

*Amplitude = \(|A|\)  (Note: Amplitude is always positive.) If \(A > 1\), then it’s a vertical stretch by a factor of \(A\). If \(0 < A < 1\), then it’s a vertical shrink by a factor of \(A\).

*If \(B > 1\), then it’s a horizontal shrink by a factor of \(1/B\). If \(0 < B < 1\), then it’s a horizontal stretch by a factor of \(1/B\).

*Period = \(\frac{2\pi}{B}\)

*Translation in horizontal direction (called the phase shift) = \(\frac{C}{B}\)

*If \(D > 0\), then it’s a vertical shift upward. If \(D < 0\) then it’s a vertical shift downward.

*If the coefficient of the function is negative, this refers to an x-axis reflection.

The graph of \( y = A \sin(Bx - C) \) completes one cycle from \(x = \frac{C}{B}\) to \(x = \frac{C}{B} + \frac{2\pi}{B}\).

The graph of \( y = A \cos(Bx - C) \) completes one cycle from \(x = \frac{C}{B}\) to \(x = \frac{C}{B} + \frac{2\pi}{B}\).

Example 2: State the transformations for:

a. \( f(x) = -2 \sin(x + 2) + 3 \)
   \[ A = -2, B = 1, C = -2, D = 3 \]
   1. \(|A| = 2 > 1\) vertical stretch by 2
   2. \(\frac{C}{B} = \frac{-2}{1} = -2 < 0\) left 2 units
   3. \(D = 3\) up
   4. x-refl.

b. \( g(x) = \cos\left(2x - \frac{\pi}{4}\right) \)
   \[ A = 1, B = 2, C = \frac{\pi}{4}, D = 0 \]
   1. \(\frac{2\pi}{B} = \frac{2\pi}{2} = \pi\) horizontal shrink by \(\frac{1}{2}\)
   2. \(\frac{C}{B} = \frac{\pi}{8} \cdot \frac{1}{2} = \frac{\pi}{16} > 0\) right by \(\frac{\pi}{16}\)

c. \( h(x) = \frac{1}{2} \sin\left(\frac{\pi}{4}x + \frac{1}{4}\right) \)
   \[ A = \frac{1}{2}, B = \frac{\pi}{4}, C = -\frac{1}{4}, D = 0 \]
   1. Vertical shrink by \(\frac{1}{2}\)
   2. \(\frac{2\pi}{B} = \frac{2\pi}{\frac{\pi}{4}} = 8\) horizontal stretch by \(8\)
   3. \(\frac{C}{B} = -\frac{1}{4} \cdot \frac{\pi}{4} = -\frac{\pi}{16} < 0\) left by \(\frac{\pi}{16}\)

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Example 3: Sketch the graph of \( f(x) = 3 \sin \left( \frac{1}{2} x \right) - 3 \). Label, with ordered pairs, any \( x \)-intercept(s), \( y \)-intercept, maxima, and minima. Then state the domain and range of the function in interval notation.

\[
A = 3 \quad B = \frac{1}{2} \quad C = 0 \quad D = -3
\]

Amplitude: \(|A| = |3| = 3\)
Period: \(\frac{2\pi}{B} = \frac{2\pi}{\frac{1}{2}} = 2\pi \cdot 2 = 4\pi\)
 Phase Shift: \(\frac{C}{B} = \frac{0}{\frac{1}{2}} = 0\)

One cycle begins at 0 & ends at \(4\pi\)
Any other transformations: 3 down

\[\text{Domain: } (-\infty, \infty) \quad \text{Range: } [-6, 0] \]
Example 4: Sketch the graph of \( f(x) = -\cos \left( 2x + \frac{\pi}{2} \right) \). Label, with ordered pairs, any \( x \)-intercept(s), \( y \)-intercept, maxima, and minima. Then state the domain and range of the function in interval notation.

\[
A = -1 \quad B = 2 \quad C = -\frac{\pi}{2} \quad D = 0
\]

Amplitude: \( |A| = |1| = 1 \)

Period: \[
\frac{2\pi}{B} = \frac{2\pi}{2} = \pi
\]

Phase shift: \[
\frac{C}{B} = \frac{-\frac{\pi}{2}}{2} = -\frac{\pi}{4} \cdot \frac{1}{2} = -\frac{\pi}{4}
\]

One cycle: \[-\frac{\pi}{4} \rightarrow \frac{\pi}{4} + \pi = \frac{\pi}{4} + \frac{4\pi}{4} = \frac{5\pi}{4}\]

Other transformer: \( x \)-reflect.

\( x \)-int: \((0,0), (\frac{\pi}{2},0)\)

\( y \)-int: \((0,0)\)

max: \((\frac{\pi}{4}, 1)\)

min: \((-\frac{\pi}{4}, -1)\)

\[
\cos \left( 2 \times \frac{\pi}{2} \right)
\]

\[
\frac{-\frac{\pi}{4} + \frac{5\pi}{4}}{2} = \frac{\pi}{4}
\]

\[
\frac{\pi}{4} + \frac{3\pi}{4} = \frac{\pi}{2}
\]

Domain: \((-\infty, \infty)\)

Range: \([-1, 1]\)
Try this one: Sketch the graph of \( f(x) = -2\cos\left(\frac{\pi x}{3}\right) \). Label, with ordered pairs, any \( x \)-intercept(s), \( y \)-intercept, maxima, and minima. Then state the domain and range of the function in interval notation.

\[ A = -2 \quad B = \frac{\pi}{3} \quad C = 0 \quad D = 0 \]

\[ |A| = 1-2 = 2 \]

Period = \( \frac{2\pi}{B} = \frac{2\pi}{\frac{\pi}{3}} = 2\pi \cdot \frac{3}{\pi} = 6 \)

Phase Shift: \( \frac{C}{B} = 0 \)

Key points: \( 0, 1.5, 3, 4.5, 6 \)

Domain: \([-\infty, \infty]\)

Try this one: Sketch the graph of \( f(x) = \sin\left(\frac{1}{4}x - \frac{\pi}{8}\right) \). Label, with ordered pairs, any \( x \)-intercept(s), \( y \)-intercept, maxima, and minima. Then state the domain and range of the function in interval notation.

\[ A = 1 \quad B = \frac{1}{4} \quad C = \frac{\pi}{8} \quad D = 0 \]

\[ |A| = |1| = 1 \]

Period = \( \frac{2\pi}{B} = 2\pi \div \frac{1}{4} = 2\pi \cdot 4 = 8\pi \)

Phase shift: \( \frac{C}{B} = \frac{\pi}{8} \div \frac{1}{4} = \frac{\pi}{2} \cdot 4 = \frac{\pi}{2} \)

Key points: \( \frac{\pi}{2}, \frac{\pi}{2} + 2\pi = \frac{5\pi}{2}, \)

\( \frac{5\pi}{2} + 2\pi = \frac{9\pi}{2}, \) \( \frac{9\pi}{2} + 2\pi = \frac{13\pi}{2} \)

\( \frac{13\pi}{2} + 2\pi = \frac{17\pi}{2} \)

Domain: \([-\infty, \infty]\)

Range: \([-1, 1]\)
Example 5: Give an equation of the form $y = A \sin(Bx - C) + D$ and $y = A \cos(Bx - C) + D$ that could represent the given graph.

$A = \frac{M - m}{2} = \frac{3 - (-1)}{2} = 2$

$D$ (half-way between max & min) = 2

Find $B$: $\frac{2\pi}{B} = (\frac{\pi}{4} - (-\frac{\pi}{4})) = \frac{2\pi}{2} = \frac{\pi}{2}$

$2\pi \neq \frac{\pi}{2}$

$BT = 4\pi \quad \Rightarrow \quad B = 4$

Phase shift: $\frac{C}{B} = -\frac{\pi}{4}$

$\frac{C}{4} = -\frac{\pi}{4} \quad \Rightarrow \quad C = -\pi$

$f(x) = 5 \sin \left(4x + \frac{\pi}{2}\right) + 2$

It may help to recall the one cycle, basic graph of:

sine: \[ \begin{array}{c}
\end{array} \]

cosine: \[ \begin{array}{c}
\end{array} \]
Try this one: Give an equation of the form \( f(x) = A \cos(Bx - C) + D \) which could be used to represent the given graph. (Note: \( C \) or \( D \) may be zero.)

\[
A = \frac{3 - (-1)}{2} = 2 \\
D = 1
\]

Try this one: Give an equation of the form \( f(x) = A \sin(Bx - C) + D \) which could be used to represent the given graph. (Note: \( C \) or \( D \) may be zero.)

\[
A = \frac{2 - (-4)}{2} = 3 \\
D = -1
\]

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The current \( I \), in amperes, flowing through an AC (alternating current) circuit at time \( t \) is given by \( I(t) = 200 \sin \left( 40 \pi t + \frac{\pi}{4} \right) \), where \( t \geq 0 \). Find the period and the horizontal shift.

a) Period: \( \frac{1}{20} \), shift \( \frac{1}{160} \) to the right.

b) Period: \( \frac{1}{40} \), shift \( \frac{1}{80} \) to the right.

c) Period: \( \frac{1}{20} \), shift \( \frac{1}{160} \) to the left.

d) Period: \( \frac{1}{20} \), shift \( \frac{1}{320} \) to the left.

e) Period: \( \frac{1}{40} \), shift \( \frac{1}{80} \) to the left.

f) None of the above.

\[
A = 200 \quad B = 40 \pi \quad C = -\frac{\pi}{4} \quad D = 0
\]

\[
P = \frac{2\pi}{B} = \frac{2\pi}{40 \pi} = \frac{1}{20}
\]

\[
h\text{-shift} = \frac{C}{B} = -\frac{\pi}{4} \div \frac{40 \pi}{1} = -\frac{1}{160}
\]

\[
\frac{1}{160} \quad \text{left}
\]
As a wave passes by, the height of the water is modeled by \( h(t) = -9 \cos \left( \frac{5\pi}{9} t \right) \).

where \( h(t) \) is the height in feet above mean sea level. Find the height of the wave. (That is, find the vertical distance between the trough and the crest of the wave.)

\[
A = |-a| = 9
\]

a) \( 9 \)
b) \( \frac{10\pi}{9} \)
c) \( 18 \)
d) \( -9 \)
e) \( \frac{5\pi}{9} \)
f) None of the above.