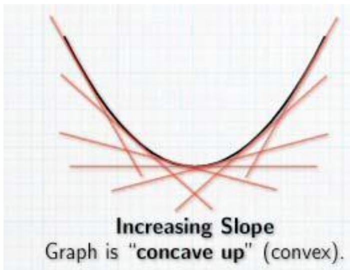
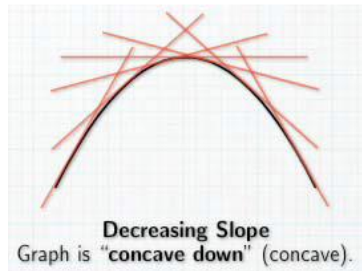
Section 3.5 – Concavity and Points of Inflection

Let f be a function that is differentiable on an open interval I.

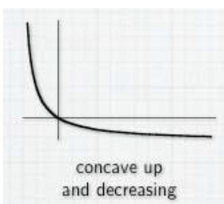
The graph of f is **concave up** if f' is increasing on I.

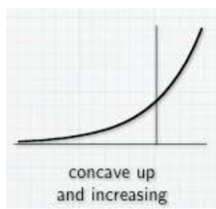
The graph of f is **concave down** if f' is decreasing on I.

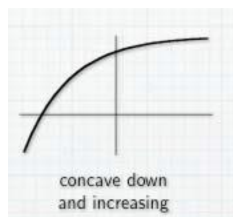


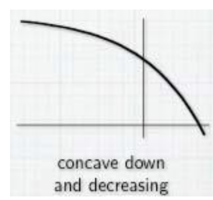


Even though both pictures indicate a local extreme value, note that that need not be the case. Here are some graphs where the functions are concave up or down without any local extreme values.

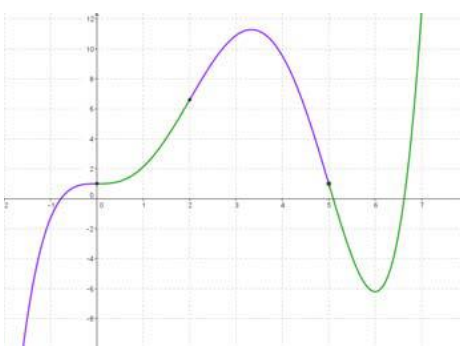








The graph of a function f is given below:



The function f is concave up over $(0, 2)$ and $(5, \infty)$.

The function f is concave down over $(-\infty, 0)$ and $(2, 5)$.

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Theorem: Let f be a function that is twice differentiable on an open interval I.

- If $f''(x) > 0$ for all x in I, then the graph of f is concave up on I.
- If $f''(x) < 0$ for all x in I, then the graph of f is concave down on I.

Let f be a function which is continuous at c and differentiable near c. The point $(c, f(c))$ is a point of inflection if the graph of f changes concavity at $x = c$.

How do we find the points of inflection?

If $(c, f(c))$ is a point of inflection, then either $f''(c) = 0$ or $f''(c)$ does not exist.

Example 1: Find the points of inflection and determine the concavity of $f(x) = 6x^5 - 20x^4 + 2x$. The domain of $f(x)$ is $(-\infty, \infty)$.

$f'(x) = 30x^4 - 80x^3 + 2$

$f''(x) = 120x^3 - 240x^2 = 120x^2(x - 2)$

Find when $f''(x) = 0$:

Find when $f''(x)$ is undefined:

$f(x)$:

$f''(x)$:

Concave Up:

Concave Down:

POI:

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Example 2: Find the points of inflection and determine the concavity of $f(x) = \frac{x}{x^2-1}$. The domain of $f(x)$ is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

$$f'(x) = \frac{-x^2-1}{(x^2-1)^2}$$

$$f''(x) = \frac{2x(x^2+3)}{(x^2-1)^3}$$

Find when $f''(x) = 0$:

Find when $f''(x)$ is undefined:

$f(x)$:

$f''(x)$:

Concave Up:

Concave Down:

POI:

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Example 3: Find the points of inflection and determine the concavity of $f(x) = x - \cos x\,, x \in [0, 2\pi]$ The domain of $f(x)$ is $(-\infty, \infty)$.

$f'(x) =$

$f''(x) =$

Find when $f''(x) = 0$:

Find when $f''(x)$ is undefined:

$f(x)$:

$f''(x)$:

Concave Up:

Concave Down:

POI:

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Example 4: Find the points of inflection and determine the concavity of $f(x) = x + x^{1/3}$. The domain of $f(x)$ is $(-\infty, \infty)$.

$f'(x) = 1 + \frac{1}{3}x^{-2/3}$

$f''(x) = \frac{-2}{9\sqrt[3]{x^5}}$

Find when $f''(x) = 0$:

Find when $f''(x)$ is undefined:

$f(x)$:

$f''(x)$:

Concave Up:

Concave Down:

POI:

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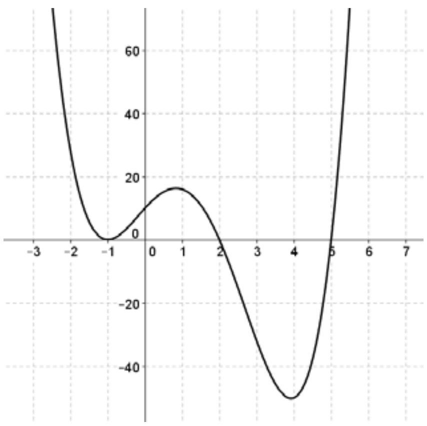
Graphs of f, f', and f''

When the graph of the second derivative is given, we can gather information about whether f' is increasing or decreasing, and whether f is concave up or down. We can also figure out the points of inflection for f.

Example 5: The graph of f''(second derivative!) of a polynomial function f is given. Determine whether each of the following statements is/are true or false.



a. The function $f(x)$ concave up over two intervals.

b. The function $f(x)$ concave down over one intervals.

c. The x-values of the points of inflection are: $x = -1, x = 2,\ x = 5$.

d. The function $f'$is increasing over $(-\infty, 2)$ and $(5, \infty)$.

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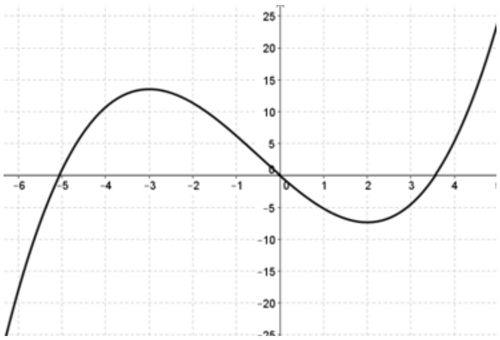
Example 6: The graph of $f'(x)$ (first derivative!) is given.



a. When is $f(x)$ concave up?

b. When is $f(x)$ concave down?

c. What are the x-values of any points of inflection of $f(x)$?

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