MATH 3336 – Discrete Mathematics
Recurrence Relations (8.1, 8.2)

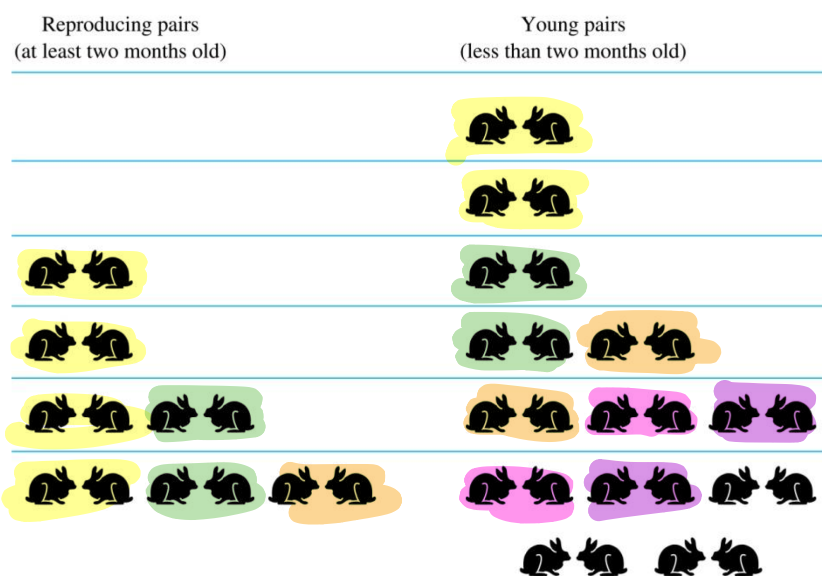
Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Rabbits and the Fibonacci Numbers

Example: A young pair of rabbits (one of each gender) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month. Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that rabbits never die.

This is the original problem considered by Leonardo Pisano (Fibonacci) in the thirteenth century.



Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8

Solution:

Let f_n be a number of rabbits after n month. We will show that $f_n, n = 1, 2, 3, \ldots$, are the terms of Fibonacci sequence.

At the end of the first month, the number of pairs of rabbits on the island is $f_1 = 1$.

Since this pair does not breed during the second month, $f_2 = 1$.

The number of pairs after n month = "old" pairs + new pairs.

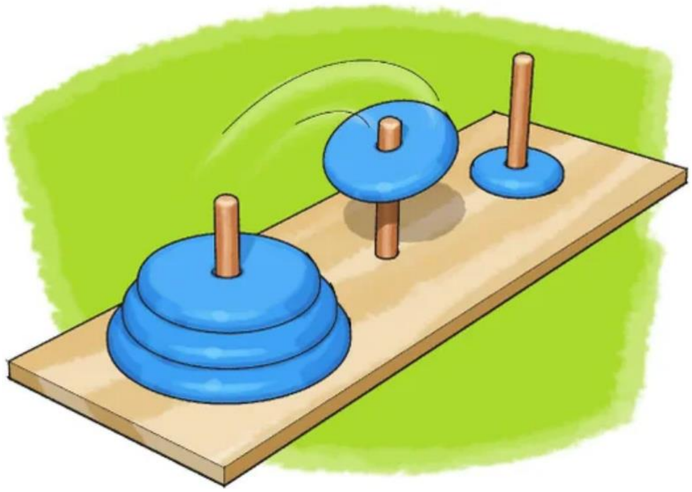
"old" = pairs that existed in month $(n-1) = f_{n-1}$

new = newborn = pairs that existed in month $(n-2) = f_{n-2}$

$$\boxed{f_n = f_{n-1} + f_{n-2} \quad n \geq 3 \qquad f_1 = 1 \quad f_2 = 1}$$

The Tower of Hanoi

In the late nineteenth century, the French mathematician Édouard Lucas invented a puzzle consisting of three pegs on a board with disks of different sizes. Initially all of the disks are on the first peg in order of size, with the largest on the bottom.



Rules: You are allowed to move the disks one at a time from one peg to another as long as a larger disk is never placed on a smaller.
Goal: Using allowable moves end up with all the disks on the second peg in order of size with largest on the bottom.

Solution:

Let H_n be the number of moves to solve the Tower of Hanoi puzzle with n disks.

Begin with n disks on Peg 1.

We can move (n-1) disks to Peg 3 with H_{n-1} moves.

Then with one move transfer the largest disk to Peg 2.

Then transfer (n-1) disks to Peg 2 with H_{n-1} moves.

$H_n = H_{n-1} + 1 + H_{n-1} = 2H_{n-1} + 1 \quad H_1 = 1$

$$H_n = 2H_{n-1} + 1 \qquad H_1 = \underline{1}$$

$$= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1$$

$$= 2^2(2H_{n-3} + 1) + 2 + 1$$

$$= 2^3 H_{n-3} + 2^2 + 2 + 1$$

$$\vdots$$

$$= 2^{n-1} H_1 + 2^{n-2} + \ldots + 2 + 1$$

$$= 2^{n-1} + 2^{n-2} + \ldots + 2 + 1$$

$$2^k H_{n-k}$$

$$n - k = 1$$

$$k = n - 1$$

Sum of geom. seq.

$$S = a\left(\frac{1 - r^n}{1 - r}\right)$$

$$= \frac{1 - 2^n}{1 - 2} = \frac{1 - 2^n}{-1}$$

$$= 2^n - 1$$

$$\boxed{H_n = 2^n - 1}$$

- There was a myth created with the puzzle. Monks in a tower in Hanoi are transferring 64 gold disks from one peg to another following the rules of the puzzle. They move one disk each day. When the puzzle is finished, the world will end.
- Using this formula for the 64 gold disks of the myth,

$$2^{64} - 1 = 18{,}446,\ 744{,}073,\ 709{,}551{,}615 \qquad \approx 4.5$$

days are needed to solve the puzzle, which is more than 500 billion years.

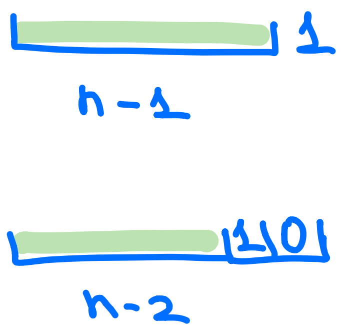
- Reve's puzzle (proposed in 1907 by Henry Dudeney) is similar but has 4 pegs. There is a well-known unsettled conjecture for the minimum number of moves needed to solve this puzzle.

Counting Bit Strings

Example: Find a recurrence relation and give initial conditions for the number of bit strings of length n without two consecutive 0s. How many such bit strings are there of length five?

Solution:

Let a_n be the number of bit strings of length n with no two consecutive 0s.



$$a_n = a_{n-1} + a_{n-2}$$

$a_1 = 2$ (0, 1)

$a_2 = 3$ (01, 10, 11)

$$f_n = f_{n-1} + f_{n-2}$$

$f_1 = 1$

$f_2 = 1$

f_1	f_2	f_3	f_4	f_5	
1,	1,	2,	3,	5,	...

2	3	5	8	11
a_1	a_2	a_3	a_4	a_5

Types of Recurrences with Constant Coefficients

- **Linear Homogeneous Recurrence Relations**

Definition: A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

, where $c_1, c_2, \dots, c_k$ are real numbers, and $c_k \neq 0$.

- **Linear Nonhomogeneous Recurrence Relations with Constant Coefficients**

  Definition: A *linear* ***non****homogeneous recurrence relation with constant coefficients* is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n)$$

where $c_1, c_2, \dots, c_k$ are real numbers, and $f(n)$ is a function not identically zero depending only on n.

The recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ is called the associated homogeneous recurrence relation.

- **Nonlinear Recurrence Relations**
  - Have nonlinear terms.
  - Hard to solve; will not discuss

Example: Which of these are linear homogeneous recurrence relations with constant coefficients (LHRRCC)? State the degree for each LHRRCC.

1. $f_n = f_{n-1} + f_{n-2}$ YES, deg = 2

2. $a_n = a_{n-1} + a_{n-2}^2$ NOT linear

3. $H_n = 2H_{n-1} + 1$ LRRCC NON homog.

4. $b_n = nb_{n-1}$ NOT CC

$c_n = 2c_{n-1} + 3c_{n-5}$ deg = 5

Solving Linear Homogeneous Recurrence Relations

Solving Linear Homogeneous Recurrence Relations of Degree Two – Two Distinct Characteristic Roots

Definition: If $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$, then deg=k

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

is the characteristic equation of a_n.

Theorem 1: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution to the recurrence relation

$a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0,1,2,\ldots$, where α_1 and α_2 are constants.

Example: What is the solution of the recurrence relation

$$a_n = \overset{c_1}{a_{n-1}} + \overset{c_2}{2}a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

Step 1: Write the characteristic equation of a recurrence relation (CERR).

If $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, then $r^2 - c_1 r - c_2 = 0$ is the characteristic equation of a_n.

$r^2 - r - 2 = 0$ $c_1 = 1$ $c_2 = 2$ $k = 2$

Step 2: Solve CERR

$(r-2)(r+1) = 0$

$r_1 = 2$ $r_2 = -1$

Step 3: Write solution in terms of αs.

$a_n = \alpha_1 (2)^n + \alpha_2 (-1)^n$

Step 4: Find α_1 and α_2 from the initial conditions.

$a_0 = \alpha_1 \cancel{(2)^0}^{1} + \alpha_2 \cancel{(-1)^0}^{1} = 2$

$a_1 = \alpha_1 (2)^1 + \alpha_2 (-1)^1 = 7$

$$\begin{array}{rl} & \alpha_1 + \cancel{\alpha_2} = 2 \\ + & 2\alpha_1 - \cancel{\alpha_2} = 7 \\ \hline & 3\alpha_1 = 9 \end{array}$$

$\alpha_1 = 3$

Sub. back: $3 + \alpha_2 = 2$

$\alpha_2 = -1$

Step 5: Write the solution.

$$\boxed{a_n = 3(2)^n - 1(-1)^n}$$

$$f_{-2}, f_{-1}, \boxed{f_0}, f_1, f_2, f_3, f_4, \ldots$$
$$-1 \quad 1 \quad \boxed{0} \quad 1 \quad 1 \quad 2 \quad 3$$

Example: Find an explicit formula for the Fibonacci numbers.

Step 1: Write the characteristic equation of a recurrence relation (CERR).

If $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, then $r^2 - c_1 r - c_2 = 0$ is the characteristic equation of a_n.

$$f_n = f_{n-1} + f_{n-2}$$
$$f_1 = 1 \quad f_2 = 1$$

$$r^2 - r - 1 = 0$$

Step 2: Solve CERR

$$ax^2 + bx + c = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 1 \quad b = -1 \quad c = -1$$
$$r = \frac{1 \pm \sqrt{(-1)^2 - 4(-1)}}{2}$$

$$r_1 = \frac{1+\sqrt{5}}{2} \quad r_2 = \frac{1-\sqrt{5}}{2}$$

Step 3: Write solution in terms of αs.

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\cancel{f_1 = 1} \quad \cancel{f_2 = 1}$$
$$f_0 = 0 \quad f_1 = 1$$

Step 4: Find α_1 and α_2 from the initial conditions.

$$f_0 = \alpha_1 \cancel{\left(\frac{1+\sqrt{5}}{2}\right)^0}^{1} + \alpha_2 \cancel{\left(\frac{1-\sqrt{5}}{2}\right)^0}^{1} = 0$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$\alpha_1 + \alpha_2 = 0 \rightarrow \alpha_2 = -\alpha_1$$

$$\alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$\alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) - \alpha_1 \left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$\alpha_1 \left(\frac{\cancel{1}+\sqrt{5} \cancel{-1}+\sqrt{5}}{2}\right) = 1$$

$$\alpha_1 \sqrt{5} = 1 \quad \alpha_1 = \frac{1}{\sqrt{5}}$$

$$\alpha_2 = -\frac{1}{\sqrt{5}}$$

Step 5: Write the solution.

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Solving Linear Homogeneous Recurrence Relations of Degree Two – One Characteristic Root of Multiplicity Two

Theorem 2: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n = 0,1,2,\ldots$, where α_1 and α_2 are constants.

Example: What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 6$?

Step 1: Write the characteristic equation of a recurrence relation (CERR).

If $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, then $r^2 - c_1 r - c_2 = 0$ is the characteristic equation of a_n.

$$r^2 - 6r + 9 = 0$$

Step 2: Solve CERR

$$(r-3)^2 = 0 \qquad r_0 = 3$$

Step 3: Write solution in terms of αs.

$$a_n = \alpha_1 (3)^n + \alpha_2 n (3)^n$$

Step 4: Find α_1 and α_2 from the initial conditions.

$a_0 = 1$
$a_1 = 6$

$$a_0 = \alpha_1 (3)^0 + \alpha_2 \cdot 0 \cdot (3)^0 = 1$$
$$a_1 = \alpha_1 (3)^1 + \alpha_2 \cdot 1 \, (3)^1 = 6$$

$$\alpha_1 = 1$$
$$3\alpha_1 + 3\alpha_2 = 6 \qquad \alpha_2 = 1$$

Step 5: Write the solution.

$$a_n = (3)^n + n(3)^n = 3^n(n+1)$$

The General Case

Theorem 3: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots c_k a_{n-k}$$

if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + a_k r_k^n$ for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

Example: Find the solution to the recurrence relation $a_n = \overset{c_1}{6} a_{n-1} - \overset{c_2}{11} a_{n-2} + \overset{c_3}{6} a_{n-3}$ with $a_0 = 2, a_1 = 5,$ and $a_2 = 15$.

$deg = 3$

Step 1: Write a characteristic equation of a recurrence relation (CERR).

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

$$r^3 - 6r^2 + 11r - 6 = 0$$

Step 2: Solve CERR.

Guess $r = 1$ $\quad (r-1)(r^2 - 5r + 6) = 0$

$$(r-1)(r-2)(r-3)$$

$$r_1 = 1 \quad r_2 = 2 \quad r_3 = 3$$

Step 3: Write solution in terms of αs.

$$a_n = \alpha_1 (1)^n + \alpha_2 (2)^n + \alpha_3 (3)^n$$

Step 4: Find αs from the initial conditions.

$a_0 = \alpha_1 \cancelto{1}{(1)^0} + \alpha_2 \cancelto{1}{(2)^0} + \alpha_3 \cancelto{1}{(3)^0} = 2$

$a_1 = \alpha_1 (1)^1 + \alpha_2 (2)^1 + \alpha_3 (3)^1 = 5$

$a_2 = \alpha_1 (1)^2 + \alpha_2 (2)^2 + \alpha_3 (3)^2 = 15$

$\alpha_1 + \alpha_2 + \alpha_3 = 2$

$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 5$

$\alpha_1 + 4\alpha_2 + 9\alpha_3 = 15$

$\alpha_1 = 1 \quad \alpha_2 = -1 \quad \alpha_3 = 2$

Step 5: Write the solution.

$$a_n = 1(1)^n - (2)^n + 2(3)^n$$

$$a_n = 1 - 2^n + 2 \cdot 3^n$$

Theorem 4: Let $c_1, c_2, \dots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

has t distinct roots $r_1, r_2, \dots, r_t$ with multiplicities $m_1, m_2, \dots, m_t$ respectively so that $m_i \geq 1$ for $i = 0, 1, 2, \dots, t$ and $m_1 + m_2 + \ldots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots c_k a_{n-k}$$

if and only if

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n$$

for $n = 0, 1, 2, \dots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_{i-1}$.

Example: Suppose that the roots of the characteristic equation of a linear homogeneous recurrence relation are 2, 2, 2, 5, 5, and 9. What is the form of the general solution?

degree = 6

root = 2 → mult. = 3

root = 5 → mult. = 2

root = 9 → mult. = 1

$$a_n = (\underbrace{\alpha_{10}}_{1} + \underbrace{\alpha_{11}}_{2} n + \underbrace{\alpha_{12}}_{3} n^2) 2^n + (\underbrace{\alpha_{20}}_{4} + \underbrace{\alpha_{21}}_{5} \cdot n) 5^n + \underbrace{\alpha_{30}}_{6} \cdot 9^n$$

Example: Find the solution to the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - \underline{a_{n-3}}$ with initial conditions $a_0 = 1, a_1 = -2$, and $a_2 = -1$.

Step 1: Write a characteristic equation of a recurrence relation (CERR).

deg 3

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

$$r^3 + 3r^2 + 3r + 1 = 0$$

Step 2: Solve CERR

$$(r+1)^3 = 0$$

$$r = -1 \quad \text{mult.} = 3$$

Step 3: Write solution in terms of αs.

$$a_n = (\alpha_{10} + \alpha_{11} n + \alpha_{12} \cdot n^2)(-1)^n$$

Step 4: Find αs from initial conditions.

$$a_0 = \alpha_{10} = 1$$

$$a_1 = (\alpha_{10} + \alpha_{11} + \alpha_{12})(-1) = -2$$

$$a_2 = \alpha_{10} + \alpha_{11} \cdot 2 + \alpha_{12} \cdot 4 = -1$$

$$\left[\begin{array}{ll} \alpha_{10} = 1 & \\ \alpha_{10} + \alpha_{11} + \alpha_{12} = 2 & \quad \alpha_{10} + 2\alpha_{11} + 4\alpha_{12} = -1 \end{array}\right]$$

Step 5: Write the solution

$$\alpha_{10} = 1 \quad \alpha_{11} = 3 \quad \alpha_{12} = -2$$

$$a_n = (1 + 3n - 2n^2)(-1)^n$$

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients.

Definition: A *linear **non**homogeneous recurrence relation with constant coefficients* is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n)$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $f(n)$ is a function not identically zero depending only on n.

The recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ is called the associated homogeneous recurrence relation.

Example: The following are linear nonhomogeneous recurrence relations with constant coefficients:

$a_n = a_{n-1} + 2^n$

$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$

$a_n = 3a_{n-1} + n3^n$

$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$

Theorem 5: If $\left\{a_n^{(p)}\right\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n)$$

then every solution is of the form $\left\{a_n^{(p)} + a_n^{(h)}\right\}$, where $\left\{a_n^{(h)}\right\}$ is a solution of the associated homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$.

Example: Find the solution of the recurrence relation $a_n = 3a_{n-1} + 2n$ with $a_1 = 3$.

$a_n = 3a_{n-1}$

$r - 3 = 0$

$r = 3$

$a_n = \alpha \cdot 3^n$

$p_n = cn + d \quad p_{n-1} = c(n-1) + d$

particular solution

$cn + d = 3(c(n-1) + d) + 2n$

$cn + d = 3(cn - c + d) + 2n$

$cn + d = 3cn - 3c + 3d + 2n$

$cn + d - 3cn + 3c - 3d - 2n = 0$

$-2cn - 2d + 3c - 2n = 0$

$2cn + 2d - 3c + 2n = 0$

$n(2c + 2) + (2d - 3c) = 0$

$2c + 2 = 0 \quad c = -1$

$2d - 3c = 0$

$2d + 3 = 0 \quad d = -3/2$

$p_n = -n - 3/2$

$\Downarrow$

$a_n = \alpha \cdot 3^n - n - 3/2$ $\qquad a_1 = 3$

$a_1 = \alpha \cdot 3 - 1 - 3/2 = 3$

$3\alpha - \frac{5}{2} = 3$

$3\alpha = 3 + \frac{5}{2}$ $\qquad 3\alpha = \frac{11}{2}$

$\alpha = \frac{11}{6}$

$$\boxed{a_n = \frac{11}{6} \cdot 3^n - n - \frac{3}{2}}$$