Discrete Mathematics
Propositional Logic

What is a proposition?

Definition: A proposition (or a statement) is a sentence that is either true or false, but not both.

Examples of Propositions:

a. Austin is the capital of Texas.
b. Texas is the largest state of the United States.
c. $1 + 0 = 1$

Examples that are NOT Propositions:

a. Watch out!
b. What time is it?
c. $x + 3 = 5$

- Letters are used to denote propositions: p, q, r, s …
- The truth value of a proposition that is always true denoted by T, the truth value of a proposition that is always false denoted by F.

New propositions (compound propositions) can be formed from existing propositions using logical operators.

Definition: Let p be a proposition. The *negation* of p, denoted by $\neg p$, the statement
“It is not the case that p.”

Examples:

a. Proposition: A triangle has three sides.
   Negation: It is not the case that triangle has three sides.
   Negation in simple English: A triangle does not have three sides.

b. Proposition: All fish can swim.
   Negation: It is not the case that all fish can swim.
   Negation in simple English: Some fish cannot swim.

c. Proposition: $2 + 3 > 5$
   Negation:

A proposition and its negation have OPPOSITE truth values!

Construct a truth table for the negation of p.

$\boldsymbol{p}$	$\boldsymbol{\neg p}$

Definition: Let p and q be propositions. The *conjunction* of p and q, denoted by $p \wedge q$, is the proposition "p and q." The conjunction is true when BOTH p and q are true and is false otherwise.

Example: Construct a truth table for the conjunction.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \wedge q}$

Example: Find the conjunction of the following propositions and determine its truth value.

a. p: All birds can fly.
   q: $2 + 3 = 5$

   Conjunction:

Definition: Let p and q be propositions. The *disjunction* of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction is false when BOTH p and q are false and is true otherwise.

Example: Construct the truth table for the disjunction.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \vee q}$
T	T	
T	F	
F	T	
F	F	

Example: Find the disjunction of the following propositions and determine its truth value.

a. p: Triangles are square.
   q: Circles are round.
   Disjunction:

Definition: Let p and q be propositions. The *exclusive* of p and q, denoted by $p \oplus q$, is the proposition "p or q, but not both." The exclusive is true when ONE of p and q is true and is false otherwise.

Example:

Students who have taken calculus or computer science can take this class.

Soup or salad comes with this entrée.

Example: Construct the truth table for the exclusive.

p	q	$p \oplus q$
T	T	
T	F	
F	T	
F	F	

Definition: Let p and q be propositions. The *conditional statement (implication)* $p \to q$ is the proposition "if p, then q."

The conditional statement $p \to q$ is false then p is true and q is false, and true otherwise.

In the conditional statement $p \to q$, p is called hypothesis and q is called conclusion.

Example: The Truth Table for the Conditional Statement $p \to q$.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

- Connection between the hypothesis and conclusion is NOT necessary.
- Think: Implication = Contract.

Example:

a. If you get 100% on the final, then you will get an A.

b. If the Moon made of cheese, then $1 + 1 = 2$.

Different Ways of Expressing $p \rightarrow q$

if p, **then** q p **implies** q
if p, q p **only if** q
q **unless** $\neg p$ q **when** p
q **if** p
q **whenever** p p **is sufficient for** q
q **follows from** p q **is necessary for** p
a necessary condition for p **is** q
a sufficient condition for q **is** p

Example:
Write the statement in the "If…, then…" form.

a. It is hot whenever it is sunny.

b. To get a good grade it is necessary that you study.

Definitions:
The proposition $q \rightarrow p$ is called *converse*.
The proposition $\neg p \rightarrow \neg q$ is called *inverse*.
The proposition $\neg q \rightarrow \neg p$ is called *contrapositive*.

Example: Write the converse, inverse, and contrapositive for the following statement.

a. If $3 \geq 5$, then $7 > 7$.

Converse:

Inverse:

Contrapositive:

b. I come to class whenever there is going to be a quiz.
(Hint: Rewrite the proposition in the "if, then" form)

Converse:

Inverse:

Contrapositive:

Definition: Let p and q be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the SAME truth values, and is false otherwise.

"if and only if" = "iff"

Example: You can drive a car if and only if your gas tank is not empty.

Example: The Truth Table for the Biconditional Statement $\leftrightarrow q$.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \leftrightarrow q}$
T	T	
T	F	
F	T	
F	F	

Expressing the Biconditional
p **is necessary and sufficient for** q
if p **then** q **, and conversely**
p **iff** q

Truth Tables for Compound Propositions
Construction of a truth table:

1. Rows
   - Need a row for every possible combination of values for the atomic propositions.
2. Columns
   - Need a column for the compound proposition (usually at far right)
   - Need a column for the truth value of each expression that occurs in the compound proposition as it is built up.

Example: Construct a truth table for $(p \vee \neg q) \rightarrow (p \wedge q)$

					$(p \vee \neg q) \rightarrow (p \wedge q)$

Equivalent Propositions

Definition: Two propositions are equivalent if they always have the same truth value.

Example: Show using a truth table that the conditional is equivalent to the contrapositive.

Precedence of Logical Operators

Operator	**Precedence**
$\neg$	**1**
$\land$	**2**
$\lor$	**3**
$\rightarrow$	**4**
$\leftrightarrow$	**5**

Example:
$p \lor q \rightarrow \neg r$ is equivalent to

If the intended meaning is $p \lor (q \rightarrow \neg r)$ then parentheses must be used.