What is a proposition?

**Definition:** A proposition (or a statement) is a sentence that is either true or false, but not both.

**Examples of Propositions:**
- Austin is the capital of Texas. \( T \)
- Texas is the largest state of the United States. \( T \)
- \( 1 + 0 = 1 \) \( T \)

**Examples that are NOT Propositions:**
- Watch out! \( F \)
- What time is it? \( F \)
- \( x + 3 = 5 \)
  - Letters are used to denote propositions: \( p, q, r, s \ldots \)
  - The truth value of a proposition that is always true denoted by \( T \), the truth value of a proposition that is always false denoted by \( F \).

New propositions (compound propositions) can be formed from existing propositions using logical operators.

**Definition:** Let \( p \) be a proposition. The *negation of \( p \)*, denoted by \( \sim p \), the statement “It is not the case that \( p \).”

**Examples:**
- Proposition: *A triangle has three sides.*
  - Negation: *It is not the case that triangle has three sides.*

- Proposition: *All fish can swim.*
  - Negation: *It is not the case that all fish can swim.*

- Proposition: *2 + 3 > 5*
  - Negation:
A proposition and its negation have **OPPOSITE truth values**!

Construct a truth table for the negation of \( p \).

\[
\begin{array}{c|c}
 p & \neg p \\
\hline
 T & F \\
 F & T \\
\end{array}
\]

**Definition:** Let \( p \) and \( q \) be propositions. The *conjunction* of \( p \) and \( q \), denoted by \( p \land q \), is the proposition “\( p \) and \( q \).” The conjunction is true when BOTH \( p \) and \( q \) are **true** and is **false** otherwise.

**Example:** Construct a truth table for the conjunction.

\[
\begin{array}{c|c|c}
 p & q & p \land q \\
\hline
 T & T & T \\
 T & F & F \\
 F & T & F \\
 F & F & F \\
\end{array}
\]

**Example:** Find the conjunction of the following propositions and determine its truth value.

a. \( p \): All birds can fly.
   \( q \): \( 2 + 3 = 5 \)

Conjunction:

\[\text{All birds can fly and } 2 + 3 = 5.\]

**False**
**Definition:** Let $p$ and $q$ be propositions. The *disjunction* of $p$ and $q$, denoted by $p \lor q$, is the proposition “$p$ or $q$.” The disjunction is false when BOTH $p$ and $q$ are false and is true otherwise.

**Example:** Construct the truth table for the disjunction.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \lor q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

**Example:** Find the disjunction of the following propositions and determine its truth value.

a. $p$: Triangles are square.
   $q$: Circles are round.

Disjunction:

**Definition:** Let $p$ and $q$ be propositions. The *exclusive* of $p$ and $q$, denoted by $p \oplus q$, is the proposition “$p$ or $q$, but not both.” The exclusive is true when ONE of $p$ and $q$ is true and is false otherwise.

**Example:**

Students who have taken calculus or computer science can take this class.

```
Disj
```

Soup or salad comes with this entrée.

```
Exc.
```
Example: Construct the truth table for the exclusive.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \oplus q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Definition: Let $p$ and $q$ be propositions. The conditional statement (implication) $p \rightarrow q$ is the proposition "if $p$, then $q$.

The conditional statement $p \rightarrow q$ is false then $p$ is true and $q$ is false, and true otherwise.

In the conditional statement $p \rightarrow q$, $p$ is called hypothesis and $q$ is called conclusion.

Example: The Truth Table for the Conditional Statement $p \rightarrow q$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \rightarrow q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

- Connection between the hypothesis and conclusion is NOT necessary.
- Think: Implication = Contract.

Example:

a. If you get 100% on the final, then you will get an A.

b. If the Moon made of cheese, then $1 + 1 = 2$. 
Different Ways of Expressing $p \rightarrow q$

- if $p$, then $q$  \quad $p$ implies $q$
- if $p, q$  \quad $p$ only if $q$
- $q$ unless $\neg p$  \quad $q$ when $p$
- $q$ if $p$
- $q$ whenever $p$  \quad $p$ is sufficient for $q$
- $q$ follows from $p$  \quad $q$ is necessary for $p$
- a necessary condition for $p$ is $q$
- a sufficient condition for $q$ is $p$

**Example:**
Write the statement in the “If..., then...” form.

a. It is hot whenever it is sunny.

\[ \text{If it is sunny, then it is hot.} \]

b. To get a good grade it is necessary that you study.

\[ \text{Studying is necessary for good grades.} \]

\[ \text{If you get good grades, then you study.} \]
Definitions:
The proposition \( q \rightarrow p \) is called \textit{converse}.
The proposition \( \neg p \rightarrow \neg q \) is called \textit{inverse}.
The proposition \( \neg q \rightarrow \neg p \) is called \textit{contrapositive}.

Example: Write the converse, inverse, and contrapositive for the following statement.

a. \( \text{If } 3 \geq 5, \text{ then } 7 > 7. \)

Converse:
\[
\text{If } 7 > 7, \text{ then } 3 \geq 5. \quad \text{True}
\]

Inverse:
\[
\text{If } 3 \leq 5, \text{ then } 7 \leq 7. \quad \text{True}
\]

Contrapositive:
\[
\text{If } 7 \leq 7, \text{ then } 3 \leq 5. \quad \text{True}
\]

b. I come to class whenever there is going to be a quiz.

(Hint: Rewrite the proposition in the “if, then” form)

If there is going to be a quiz, then I come to class.

Converse:
If I come to class, then there is going to be a quiz.

Inverse:
If there is not going to be a quiz, then I do not come to class.

Contrapositive:
If I do not come to class, then there is not going to be a quiz.
Definition: Let $p$ and $q$ be propositions. The biconditional statement $p \iff q$ is the proposition “$p$ if and only if $q$.” The biconditional statement $p \iff q$ is true when $p$ and $q$ have the same truth values, and is false otherwise.

“if and only if” = “iff”

Example: You can drive a car if and only if your gas tank is not empty.

Example: The Truth Table for the Biconditional Statement $\iff q$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \iff q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
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<td>F</td>
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<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Expressing the Biconditional
$p$ is necessary and sufficient for $q$
if $p$ then $q$, and conversely
$p$ iff $q$

Truth Tables for Compound Propositions
Construction of a truth table:
1. Rows
   - Need a row for every possible combination of values for the atomic propositions.
2. Columns
   - Need a column for the compound proposition (usually at far right)
   - Need a column for the truth value of each expression that occurs in the compound proposition as it is built up.
**Example:** Construct a truth table for \((p \lor \neg q) \rightarrow (p \land q)\)

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(p)</td>
<td>(q)</td>
<td>(\neg q)</td>
<td>(p \lor \neg q)</td>
<td>(p \land q)</td>
<td>((p \lor \neg q) \rightarrow (p \land q))</td>
</tr>
<tr>
<td>(T)</td>
<td>(T)</td>
<td>(F)</td>
<td>(T)</td>
<td>(T)</td>
<td>(T)</td>
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<tr>
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<td>(F)</td>
<td>(F)</td>
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<td>(T)</td>
<td>(F)</td>
<td>(F)</td>
</tr>
</tbody>
</table>

**Equivalent Propositions**

**Definition:** Two propositions are equivalent if they always have the same truth value.

**Example:** Show using a truth table that the conditional is equivalent to the contrapositive.

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(p)</td>
<td>(q)</td>
<td>(p \rightarrow q)</td>
<td>(\neg q)</td>
<td>(\neg p)</td>
<td>(\neg q \rightarrow \neg p)</td>
</tr>
<tr>
<td>(T)</td>
<td>(T)</td>
<td>(T)</td>
<td>(F)</td>
<td>(F)</td>
<td>(T)</td>
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<td>(T)</td>
<td>(T)</td>
</tr>
</tbody>
</table>
Precedence of Logical Operators

<table>
<thead>
<tr>
<th>Operator</th>
<th>Precedence</th>
</tr>
</thead>
<tbody>
<tr>
<td>¬</td>
<td>1</td>
</tr>
<tr>
<td>∧</td>
<td>2</td>
</tr>
<tr>
<td>∨</td>
<td>3</td>
</tr>
<tr>
<td>→</td>
<td>4</td>
</tr>
<tr>
<td>↔</td>
<td>5</td>
</tr>
</tbody>
</table>

Example:

$p \lor q \land \lnot r$ is equivalent to $(p \lor q) \rightarrow \lnot r$

If the intended meaning is $p \lor (q \rightarrow \lnot r)$ then parentheses must be used.