## Discrete Mathematics
## Predicates and Quantifiers

### Predicates

Propositional logic is not enough to express the meaning of all statements in mathematics and natural language.

**Examples:**
Is "$x > 1$" True or False?

Is "$x$ is a great tennis player" True or False?

**Predicate Logic**

- Variables: $x$, $y$, $z$, etc.
- Predicates: $P(x)$, $Q(x)$, etc.
- Quantifiers: Universal and Existential.
- Connectives from propositional logic carry over to predicate logic.

A **predicate** $P(x)$ is a declarative sentence whose truth value depends on one or more variables.

$P(x)$ is also said to be the value of the **propositional function** $P$ at $x$.

$P(x)$ becomes a **proposition** when a value of $x$ is assigned from the domain $U$.

**Examples (Propositional Functions):**

1. Let $P(x)$ be "$x \geq 1$." Determine the truth value of

   a. $P(2)$

   b. $P(-2) \rightarrow P(1)$

2. Let $R(x, y, z)$ be "$x + y = z$." Find these truth values:

   a. $R(2, -1, 5)$

   b. $R(x, 3, z)$

## Quantifiers

We need *quantifiers* to express the meaning of English words including *all* and *some*:

- "All students in this class are computer science majors"
- "There is a math major student in this class"

The two most important quantifiers are:

- *Universal Quantifier,* "For all," symbol: ∀
- *Existential Quantifier*, "There exists," symbol: ∃

We write as in $\forall x\ P(x)$ and $\exists x\ P(x)$.

- $\forall x\ P(x)$ asserts $P(x)$ is true for <u>every</u> $x$ in the *domain*.

  If $= \{x_1, x_2, \ldots, x_n\}$, then $\forall x\ P(x) = P(x_1) \wedge P(x_2) \ldots \wedge P(x_n)$.

- $\exists x\ P(x)$ asserts $P(x)$ is true for <u>some</u> $x$ in the *domain*.

  If $= \{x_1, x_2, \ldots, x_n\}$, then $\exists x\ P(x) = P(x_1) \vee P(x_2) \ldots \vee P(x_n)$.

**Examples:**

1. Let $P(x)$: "$x > -x$" with the domain of all positive real numbers.
   Find the truth value of $\forall x\ P(x)$.

2. Let $P(x)$: "$x > -x$" with the domain of all real numbers.
   Find the truth value of $\forall x\ P(x)$.

- The truth value of $\exists x\ P(x)$ and $\forall x\ P(x)$ depends BOTH on the propositional function $P(x)$ and on the domain $U$.

| ***Quantifiers*** | | |
|---|---|---|
| ***Statement*** | ***When True?*** | ***When False?*** |
| $\forall x\ P(x)$ | $P(x)$ is true for every $x$. | There is an x for which $P(x)$ is false. |
| $\exists x\ P(x)$ | There is an x for which $P(x)$ is true. | $P(x)$ is false for every $x$. |

**Example:** Suppose the domain of the propositional function $P(x): x^2 \leq x$ consists of $\{1, 2, 3\}$. Write out each of the following propositions using conjunction or disjunction and determine its truth value.

1. $\forall x\, P(x)$
2. $\exists x\, P(x)$

An element for which $P(x)$ is false is called a **counterexample of** $\forall x\, P(x)$

**Precedence of Quantifiers**

The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.

**Example:** $\forall x\, P(x) \lor Q(x)$ means $(\forall x\, P(x)) \lor Q(x)$. $\forall x\, (P(x) \lor Q(x))$ means something different.

**Negating Quantifiers**

**De Morgan laws for quantifiers** (the rules for negating quantifiers) are:

$$\neg\forall x\, P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x\, P(x) \equiv \forall x \neg\, P(x)$$

**Example:** Express each of these statements using quantifiers. Then form a negation of the statement, so that no negation is left of a quantifier. Next, express the negation in simple English.

1. “Some old dogs can learn new tricks.”

2. “Every bird can fly.”

3. $\forall x(x^2 > x)$

**Translating from English into Logical Expressions**

**Examples:** Translate the statements into the logical symbols. Let $x$ be in set of all students in this class.

1. Someone in your class can speak Hindi.
2. Everyone in your class is friendly.
3. There is a student in your class who was not born in California.

$H(x) \ = \text{“ } x\ speaks\ Hindi\text{”}, F(x) \ = \text{“ } x\ is\ friendly, \text{” } C(x) = \text{“ } x\ was\ born\ in\ California.\text{”}$

**Example:** Translate the following sentence into predicate logic and give its negation:

"Every student in this class has taken a course in Java."

**Solution**:

First, decide on the domain $U$!

**Solution 1**: If $U$ is all students in this class, define a propositional function J($x$) denoting "x has taken a course in Java" and translate as

**Solution 2**: But if $U$ is all people, also define a propositional function S(x) denoting "x is a student in this class" and translate as

**Example:** Translate the following sentence into predicate logic:

"Some student in this class has taken a course in Java."

**Solution**:

First, decide on the domain $U$!

**Solution 1**: If $U$ is all students in this class, translate as

**Solution 2**: But if $U$ is all people, then translate as