Discrete Mathematics
Sets

Definition: A *set* is an unordered collection of distinct objects.
Examples:

a. the students in this class
b. the chairs in this room
c. counting numbers

Definition: The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.
Notation:

- The notation $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, write $a \notin A$.

Ways to describe a set

1. Roster method
    - List all elements of a set
    - Order NOT important
    - Each distinct object is either a member or not; listing more than once does not change the set.
    - Ellipses (...) may be used to describe a set without listing all of the members when the pattern is clear.

$$A = \{\overset{2+2}{2, 4}, \ldots\}$$

$$2^1, 2^2, 2^3$$

Example: $S = \{a, b, c, d\}$

Usually elements in a set have a common property but this is NOT necessary.

Example: $C = \{\text{I, love, math, 10, Houston}\}$

2. Set builder notation
    - Specify a property that all elements in a certain set must have.

such that

Example: $B = \{x \mid x \textit{ is an even positive integer less than } 12\}$ ← Set builder

$B = \{2, 4, 6, 8, 10\}$ ← Roster notation

Some important sets

$\mathbb{N} = \textit{natural numbers} = \{0, 1, 2, 3 \ldots\}$

$\mathbb{Z} = \textit{integers} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$\mathbb{Z}^+ = \textit{positive integers} = \{1, 2, 3, \ldots\}$

Zahl

$\mathbb{R}$ = set of *real numbers*

$\mathbb{R}^+$ = set of *positive real numbers*

$\mathbb{C}$ = set of *complex numbers.*

$\mathbb{Q}$ = set of *rational numbers*

$\frac{p}{q}$

Example: $B = \{x \mid x \text{ is an even positive integer less than } 12\}$

$B = \{x \mid 0 < x < 12,\ x = 2k,\ k \in \mathbb{Z}\}$

Example: $\{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$

Definition: Two sets are equal if and only if they have the same elements.

Notation: $A = B$ *iff* $\forall x\ (x \in A \leftrightarrow x \in B)$

$x \in A \rightarrow x \in B$

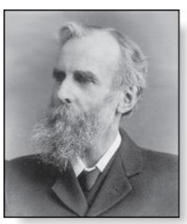
$x \in B \rightarrow x \in A$

Example: $A = \{red, blue, black\}$, $B = \{black, red, blue\}$ $A = B$

Example: $A = \{red, red, red, red\}$, $B = \{red\}$ $A = B$

Venn Diagrams

Sets can be represented graphically using Venn diagrams. Venn diagrams are often used to show relations between sets.



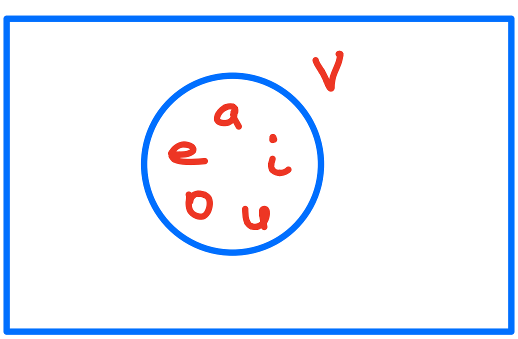
John Venn (1834-1923)
Cambridge, UK

The *universal set U* is the set containing everything currently under consideration.

- Sometimes implicit
- Sometimes explicitly stated.
- Contents depend on the context.

The *empty (null) set* is the set with no elements. Symbolized $\emptyset$, but $\{\}$ also used.

Example: Draw a Venn diagram that represents $V = \{a, e, i, o, u\}$.



$U =$ English alphabet

Example: $A = \{x \in \mathbb{N} \,|x < 0\}$ $= \phi = \{\}$

Definition: A set with one element is called a singleton set.

Example: $A = \{x \in \mathbb{N}|x \leq 0\}$ $= \{0\}$

Definition: The set A is a *subset* of B if and only if every element of A is also an element of B.

A is a *strict (proper) subset* of B if A is a subset of B and A is not equal to B.

Notation: $A \subseteq B$ and $A \subset B$ respectively.

Example: $\{1, 2\} \subset \{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4\}$

- To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.
- To show that $A \nsubseteq B$, find a single $x \in A$ such that $x \notin B$.
- For every set S, $\emptyset \subset S$ and $S \subseteq S$.

Prove: For every nonempty set S, $\emptyset \subset S$.

Proof:

If $\underbrace{x \in \phi}_{\text{False}}$, then $x \in S$. True

$\overset{F}{p} \to q$

$F \to T$ True

$F \to F$ True

Definition: Given a set A, cardinality (size) of A is the number of elements in A.

Notation: $|A|$

Definition: The power set is the set of all subsets of a given set, and is denoted by $\mathcal{P}(A)$.

If a set has n elements, its power set contains 2^n elements.

$\times$ $\{3\}$ is not a ~~subset~~ element of $\mathcal{P}(A)$

Try this:

$A = \{1, \{2\}, \{3,4\}\}$, $B = \{1, 2, 3, 4\}$

1. True or False?
   - a. $1 \in A$ T
   - b. $2 \in A$ F $\{2\} \in A$
   - c. $\{3\} \subset A$ F
   - d. $2 \in B$ T
   - e. $\{3\} \subset B$ T
   - f. $\emptyset \in A$ F $\emptyset \subset A$ T
   - g. $\emptyset \subset B$ T
   - h. $A \subseteq B$ False, $\{3,4\} \in A$, but $\{3,4\} \notin B$

2. **Find**
   - a. $|A| = 3$
   - b. $|B| = 4$
   - c. $\mathcal{P}(A)$ $\mathcal{P}(A) = \{\emptyset, \{1\}, \{\{2\}\}, \{\{3,4\}\}, \{1, \{2\}\}, \{1, \{3,4\}\}, \{\{2\}, \{3,4\}\}, \{1, \{2\}, \{3,4\}\}\}$
   - d. $|\emptyset| = 0$
   - e. $|\{\emptyset\}| = 1$, contains 1 element $\emptyset$
   - f. $\mathcal{P}(\emptyset) =$
   - g. $\mathcal{P}(\{\emptyset\})$

$$\mathcal{P}(\emptyset) = \{\emptyset\} = \{\{\}\} \qquad |\mathcal{P}(\emptyset)| = 2^0 = 1$$

$$\mathcal{P}(\{\emptyset\}) = \{\{\}, \{\{\}\}\} = \{\emptyset, \{\emptyset\}\} \qquad |\{\emptyset\}| = 1 \qquad |\mathcal{P}(\{\emptyset\})| = 2^1 = 2$$

$\mathcal{P}(A) = \{ \emptyset, \qquad , A \}$

$2^{|A|}$ $\qquad A \subseteq A$

$A = \{ 1, \{2\}, \{3,4\} \}$

$\{ 1, \{2\} \}$ $\qquad \boxed{\{ \{2\}, \{3,4\} \}}$

$\mathcal{P}(A) = \{ \emptyset, \{1\}, \{\{2\}\}, \dots$

$\emptyset \subset A$

$\{1\} \subset A$

$\{\{2\}\} \subset A$

$C = \{a, b, c\}$

$\mathcal{P}(C) = \{ \emptyset, \{a\}, \{b\}, \{c\},$
$\{a,b\}, \{a,c\}, \{b,c\},$
$\{a,b,c\} \}$

$\{a\} \neq \{a,b\}$

$\{a\} \subset C$

$\{a,b\} \subset C$