Discrete Mathematics
Functions

Definition: A *function* f from a set A to a set B, denoted $f: A \rightarrow B$ is a well-defined rule that assigns each element of A to exactly one element of B.

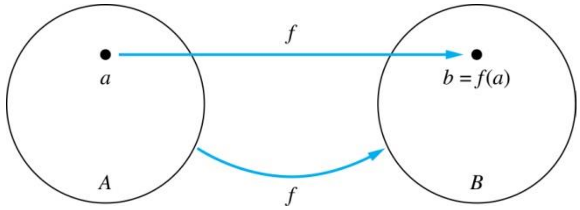
We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

Example:

Recall: Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.

We can define a function $f: A \rightarrow B$ as a subset of the Cartesian product $A \times B$.

Given a function $f: A \to B$**:**



- We say *f maps A* to *B or f* is a *mapping* from *A* to *B*.
- *A* is called the *domain* of *f*.
- *B* is called the *codomain* of *f*.
- If *f*(*a*) = *b*,
    - then *b* is called the *image* of *a* under *f*.
    - *a* is called the *preimage* of *b.*
- The range of *f* is the set of all images of points in **A** under *f*. We denote it by *f*(***A***).
- Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.

Example: Let $f: A \to B$, where $A = \{a, b, c, d\}$, $B = \{x, y, z\}$, and $f(a) = z$, $f(b) = y$, $f(c) = z$, $f(d) = z$.

a. $f(a) =?$

b. The image of d is?

c. The domain of f is?

d. The codomain of f is?

e. The preimage of y is?

f. $f(A) = ?$

g. The preimage(s) of z is (are)?

h. The range of f is?

Definition: A function f is said to be *one-to-one*, or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one.

Example:

Definition: A function f from A to B is called *onto* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called a *surjection* if it is *onto*.

Example:

Definition: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).

Example:

Suppose that $f: A \to B$.

To show that *f* is injective: Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$.

To show that *f* is *not* injective: Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that *f* is surjective: Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that *f* is *not* surjective: Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Example: Let f be the function from $\{a, b, c, d\}$ to $\{1,2,3\}$ defined by $f(a) = 3, f(b) = 2, f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Example: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Example: Determine if each function is a bijection.

a. $f(x) = 2x + 1$

b. $f(x) = x^2 + 1$

Definition: Let f be a bijection from A to B. Then the *inverse* of f, denoted f^{-1}, is the function from B to A defined as $f^{-1}(y) = x$ iff $f(x) = y$.

No inverse exists unless f is a bijection. Why?

Example: Let f be the function from $\{a, b, c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Example: Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be such that $f(x) = 2x + 1$. Is f invertible, and if so, what is its inverse?

Example: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be such that $f(x) = |x|$. Is f invertible, and if so, what is its inverse?

Definition: Let $f: B \to C$, $g: A \to B$. The *composition of* f *with* g, denoted $f \circ g$ is the function from A to C defined by $(f \circ g)(x) = f(g(x))$.

Example: $f(x) = x^2$ and $g(x) = 2x + 1$. Find $(f \circ g)(x)$ and $g \circ f(x)$.

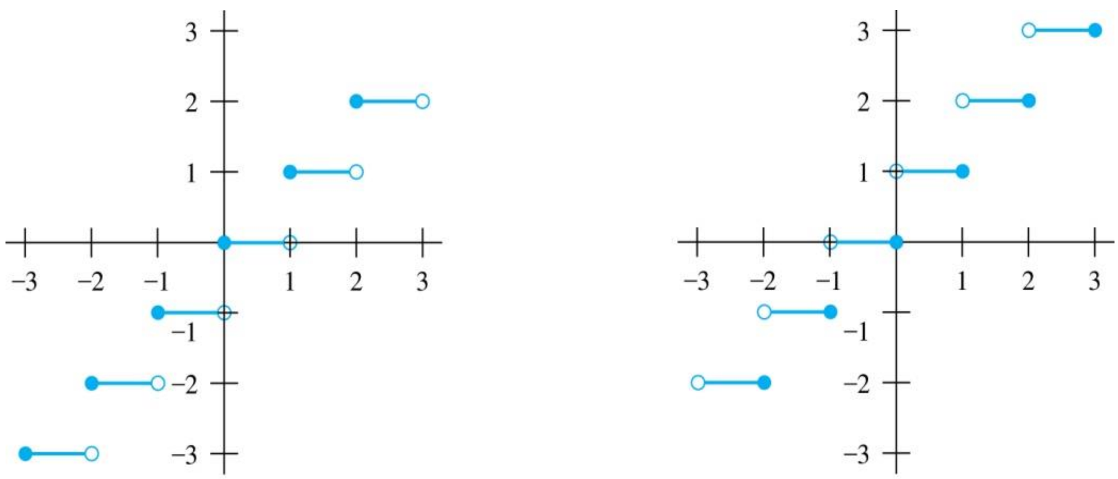
Example: Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of f and g, and what is the composition of g and f.

Some important functions

- The *floor* function, denoted $\lfloor x \rfloor$ is the largest integer less than or equal to x.
- The *ceiling* function, denoted $\lceil x \rceil$is the smallest integer greater than or equal to x.

Example:



(a) $y = \lfloor x \rfloor$ (b) $y = \lceil x \rceil$

TABLE 1 Useful Properties of the Floor and Ceiling Functions. (*n* is an integer, ***x*** is a real number)	
(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

- Factorial function

Definition: $f: \mathbb{N} \to \mathbb{Z}^+$**,** denoted by $f(n) = n!$ is the product of the first n positive integers.
$f(n) = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$. $f(0) = 0! = 1$

Example: Find f(2), f(3), f(4).