Discrete Mathematics
Sequences and Summations

Definition: A *sequence* is a function from a subset of the integers (usually either the set $\{0, 1, 2, 3, 4, \dots\}$ or $\{1, 2, 3, 4, \dots\}$) to a set S.

We use the notation a_n to denote the image of the integer n. We call a_n *a term* of the sequence.

Example: Consider the sequence $\{a_n\}$, where $a_n = \frac{1}{n}$. List the terms of this sequence beginning with a_1.

Geometric Progression

Definition: A *geometric progression* is a sequence of the form: $a, ar, ar^2, ar^3, \dots, ar^n, \dots$ where the *initial term* a and the *common ratio* r are real numbers.

Remark: A geometric progression is a discrete version of the exponential function

$$f(x) = ar^x$$

Examples: Write geometric progression with the following parameters:

a. $a = 1 \; and \; r = -1$

b. $a = 2 \; and \; r = 5$

c. $a = 6 \; and \; r = 1/3$

Arithmetic progression

Definition: An *arithmetic progression* is a sequence of the form: $a, a+d, a+2d, a+3d, \ldots, a+nd, \ldots$ where the *initial term* a and the *common difference* d are real numbers.

Remark: An arithmetic progression is a discrete analogue of the linear function

$$f(x) = dx + a$$

Example: Write arithmetic progression with the following parameters:

a. $a = -1$ and $d = 4$

b. $a = 7 \text{ and } d = -3$

c. $a = 1$ and d = 2

Recurrence Relations

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Example: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1,2,3, \ldots$ and suppose that $a_0 = 2$. What are a_1, a_2, a_3?

Example: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2,3,4,\ldots$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

Fibonacci sequence

Definition: Define the *Fibonacci sequence*, $f_0, f_1, f_2, f_3, \ldots$ by:
Initial Conditions: $f_0 = 0, f_1 = 1$
Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

Example: Write the first six terms of the Fibonacci sequence.

$f_2 = f_1 + f_0 =$

$f_3 = f_2 + f_1 =$

$f_4 = f_3 + f_2 =$

$f_5 = f_4 + f_3 =$

$f_6 = f_5 + f_4 =$

Solving recurrence relation

- Finding a formula for the *n*th term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.
- Such a formula is called a *closed formula*.
- Various methods for solving recurrence relations will be covered later.
- Here we illustrate by example the method of iteration in which we need to guess the formula. The guess can be proved correct by the method of Mathematical Induction.

Example: Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \dots$.

Answer the same question where $a_n = 2^n$ and $a_n = 5$.

Note: There may be more than one solution to a recurrence relation.

Example (Iterative solution): Solve the recurrence relation and initial condition**:**
$a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots$ and $a_1 = 2$.

Forward substitution:

Backward substitution:

Note: When we use iteration, we essentially guess a formula for a term of a sequence. To prove our guess is correct, we need to use Mathematical Induction.

More on solving recurrence relations
Given a few terms of a sequence, try to identify the sequence. Conjecture a formula, recurrence relation, or some other rule.

Some questions to ask?
- Are there repeated terms of the same value?
- Can you obtain a term from the previous term by adding an amount or multiplying by an amount?
- Can you obtain a term by combining the previous terms in some way?
- Are they cycles among the terms?
- Do the terms match those of a well-known sequence?

Example: Find formulae for the sequences with the following first five terms:

a. $1, 1/2, 1/4, 1/8, 1/16$

b. 1, 3, 5, 7, 9

c. 1, -1, 1, -1, 1

Some Useful Sequences	
nth Term	**First 5 Terms**
n^2	1, 4, 9, 16, 25, …
n^3	1, 8, 27, 64, 125, …
n^4	1, 16, 81, 256, 625, …
2^n	2, 4, 8, 16, 32, …
3^n	3, 9, 27, 81, 243, …
$n!$	1, 2, 6, 24, 120, …
f_n	1, 1, 2, 3, 5, …

Summations

Consider sum of the terms $a_m, a_{m+1}, \dots, a_n$ of the sequence $\{a_n\}$.
We use notation $\sum_{j=m}^{n} a_j$, $\sum_{m \le j \le n} a_j$ to represent $a_m + a_{m+1} + \cdots + a_n$.
j is called the *index of summation*, m is the *lower limit*, and n is the *upper limit*.

Examples:

a. $\sum_1^5 j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$

b. $\sum_1^\infty \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

Some Useful Summation Formulae	
Sum	**Closed Form**
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k+0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k+0}^{\infty} kx^k, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Example: If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r - 1} & r \neq 1 \\ (n+1)a & r = 1 \end{cases}$$

Proof: