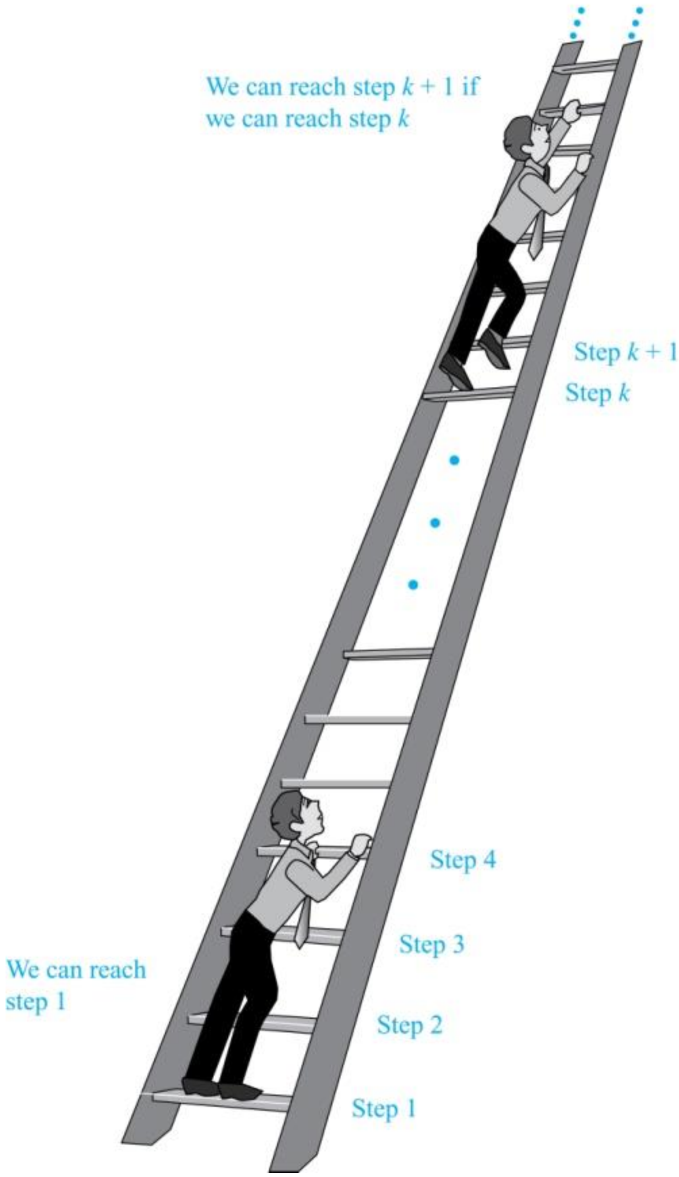
MATH 3336 – Discrete Mathematics
Mathematical Induction (5.1)



(from Discrete Mathematics and Its Applications by K. Rosen)

Suppose we have an infinite ladder:

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

Principle of Mathematical Induction: To prove that $P(n)$ is true for all positive integers n, we complete these steps:

- *Basis Step*: Show that $P(1)$ is true.
- *Inductive Step*: Show that $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

To complete the inductive step, assuming the *inductive hypothesis* that $P(k)$ holds for an arbitrary integer k, show that must $P(k + 1)$ be true.

Important points about Mathematical Induction

- Mathematical induction can be expressed as the rule of inference where the domain is the set of positive integers.

$$\big(P(1) \wedge \forall k\ (P(k) \rightarrow P(k+1))\big) \rightarrow \forall n\ P(n)$$

- In a proof by mathematical induction, we do not assume that $P(k)$ is true for all positive integers! We show that if we assume that if $P(k)$ is true, then $P(k+1)$ must also be true.
- Proofs by mathematical induction do not always start at the integer 1. In such a case, the basis step begins at a starting point b where b is an integer. We will see examples of this soon.
- Mathematical Induction cannot be used to find new theorems and does not give insights on why a theorem works.

Example: Show that if n is a positive integer, then $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Basis step: $P(\ )$

Inductive step: $P(k) \rightarrow P(k+1)$

Conclusion: By the principle of induction, the statement is true for all $n \in \mathbb{Z}^{+}$.

Example: Conjecture a formula for the sum of the first n positive odd integers. Then prove the conjecture using mathematical induction.

Basis step: $P(\ )$

Inductive step: $P(k) \rightarrow P(k+1)$

Conclusion: By the principle of induction, the statement is true for all odd positive n.

Example: Use mathematical induction to prove $2^n < n!$ For every integer n with $n \geq 4$.

Basis step: $P(\ )$

Inductive step: $P(k) \rightarrow P(k+1)$

Conclusion: By the principle of induction, the statement is true for all integer $n \geq 4$.

Example: Prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

Basis step: $P(\ \ )$

Inductive step: $P(k) \rightarrow P(k+1)$

Conclusion: By the principle of induction, the statement is true for all $n \in \mathbb{Z}^+$.

Example: Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer n.

Basis step: $P(\ )$

Inductive step: $P(k) \rightarrow P(k+1)$

Conclusion: By the principle of induction, the statement is true for all $n \in \mathbb{Z}^{+}$.

Example: Use mathematical induction to show that if S is a finite set with n elements, where n is a nonnegative integer, then S has 2^n subsets.

Basis step: $P(\ \ )$

Inductive step: $P(k) \rightarrow P(k+1)$

Conclusion: By the principle of induction, the statement is true for all nonnegative integers.