MATH 3336 – Discrete Mathematics
Combinations and Permutations (6.3)

Permutations

Definition: A *permutation* of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an *r-permutations.*

Notation: The number of *r-permutations* of a set with n elements is denoted by $P(n,r)$.

$P(n,n)$

Example: Let $S = \{1,2,3\}$.
- The ordered arrangement 3, 1, 2 is a permutation of S.
- The ordered arrangement 3, 2 is a 2-permutation of S.

Write all 2-permutations of $S = \{1,2,3\}$.

1,2 1,3 2,3 $P(3,2) = 6$

2,1 3,1 3,2

Theorem: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

Proof: Use the product rule

1 2 3 __ __ . . . r-1 r

n choices n-1 choices n-r+1 choices

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1)$$

■.

Corollary: If n and r are integers with $1 \leq r \leq n$, then

$$P(n,r) = \frac{n!}{(n-r)!}$$

.

$$\frac{n!}{(n-r)!} = \frac{n(n-1)(n-2)\ldots(n-r+1)\cancel{(n-r)\ldots 3\cdot 2\cdot 1}}{\cancel{(n-r)(n-r-1)\ldots 3\cdot 2\cdot 1}}$$

$$= n(n-1)(n-2)\ldots(n-r+1) = P(n,r)$$

Solving Counting Problems by Counting Permutations

Example: How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

$$P(100,3) = \frac{100!}{(100-3)!} = \frac{100!}{97!} = \frac{\cancel{1\cdot 2\cdots 97}\cdot 98\cdot 99\cdot 100}{\cancel{1\cdot 2\cdots 97}}$$

$$= 98\cdot 99\cdot 100 = 970{,}200$$

$100\cdot 99\cdot 98$

Example: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

The first city is chosen.

Recall: $0! = 1$

$$P(7,7) = \frac{7!}{(7-7)!} = \frac{7!}{0!} = 7!$$

Example: How many permutations of the letters $\boxed{ABC}DEFGH$ contain the string $\boxed{ABC}$?

1 23456

$$P(6,6) = 6! = 720$$

D E F $\boxed{ABC}$ H G

Combinations

Definition: An *r-combination* of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.

$\binom{n}{r}$

Notation: The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$. This notation is also used and is called a *binomial coefficient*.

Example: Let S be the set $\{1, 2, 3\}$. Then $\{1, 2\}$ is a 2-combination from S. It is the same as $\{2, 1\}$ since the order listed does not matter.

Find $C(3,2)$. $= 3$

$\{1,2\}$ $\{1,3\}$ $\{2,3\}$

Theorem: The number of r-combinations of a set with n elements, where $0 \leq r \leq n$, equals

$$C(n, r) = \frac{n!}{r!\,(n-r)!}$$

Proof: By product rule

$$P(n,r) = C(n,r) \cdot P(r,r)$$

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{(n-r)!} \div \frac{r!}{\cancel{(r-r)}!} \; 1$$

$$= \frac{n!}{r!\,(n-r)!}$$

Solving Counting Problems by Counting Combinations

Example: How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?

$$C(52,5) = \frac{52!}{5!\,47!} = 2,598,960$$

$$C(52,47) = \frac{52!}{47!\,5!}$$

Corollary: Let n and r be nonnegative integers with $r \leq n$. Then $C(n,r) = C(n, n-r)$.

Proof: $C(n,r) = \dfrac{n!}{r!\,(n-r)!}$

$$C(n,n-r) = \frac{n!}{(n-r)!\,(\underbrace{n-(n-r)}_{\cancel{n}-\cancel{n}+r})!} = \frac{n!}{(n-r)!\,r!}$$

$\blacksquare$.

Example: Find the following values:

$C(n,1) =$ n

$C(n,0) =$ 1

$C(n,n) =$ 1

$C(n,n-1) =$ n

Example: How many ways are there to select six players from a 10-member tennis team to make a trip to a match at another school?

$$C(10,6) = \frac{10!}{6!\,4!} = \frac{\cancel{1\cdot 2\cdots 6}\cdot 7\cdot \cancel{8}\cdot \overset{3}{\cancel{9}}\cdot 10}{\cancel{1\cdot 2\cdots 6}\ 1\cdot \cancel{2}\cdot \cancel{3}\cdot \cancel{4}}$$

$$= 210$$

Example: A judge has a jury pool of 40 people that contains 22 women and 18 men. She needs a jury of 12 people.

40

22W 18M

6W 6M

a. How many juries can be made?

$C(40,12)$

b. How many juries contain 6 women and 6 men?

$C(22,6)\cdot C(18,6)$

Example: A customer at a fruit stand picks a sample of 7 oranges at random from a crate containing 35 oranges of which 5 are rotten.

35

5R 30G

4R 3G

5R 2G

a. How many selections can be made?

$C(35,7)$

b. How many selections contain 4 rotten?

$C(5,4)\,C(30,3)$
$= 5\,C(30,3)$

c. How many selections contain at least 4 rotten? 4R or 5R

$C(5,4)\,C(30,3) + C(5,5)\,C(30,2)$
$= 5\,C(30,3) + C(30,2)$

d. How many selections contain at most 4 rotten? 0R, 1R, 2R, 3R, 4R

5R

$C(35,7) - C(30,2)$

e. How many selections contain at least 1 rotten? 1R, 2R, 3R, 4R, 5R

0R

$C(35,7) - C(30,7)$