Lecture 19 Section 10.5 Indeterminate Form (0/0) Section 10.6 Other Indeterminate Forms (∞/∞), $(0\cdot\infty)$, $\cdots$

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1 Indeterminate Forms

1.1 Indeterminate Form (0/0)

Example 1. **What is the Indeterminate Form (0/0)?**

$$\lim_{x\to 2}\frac{x-2}{x^2-4}=\lim_{x\to 2}\frac{x-2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{x+2}=\frac{1}{2+2}=\frac{1}{4}$$

$$(\textit{L'Hôpital's Rule})=\lim_{x\to 2}\frac{f'(x)}{g'(x)}=\lim_{x\to 2}\frac{1}{2x}=\frac{1}{2\cdot 2}=\frac{1}{4}\quad(\textit{Amazing!!})$$

As $x\to 2$, $f(x)=x-2\to 0$ and $g(x)=x^2-4\to 0$, but the quotient

$$\frac{f(x)}{g(x)}=\frac{x-2}{x^2-4}\to\frac{0}{0}.$$

We *can not determine a value for "$\frac{0}{0}$" without some extra work!*

If $\lim_{x\to a}f(x)=\lim_{x\to a}g(x)=0$, then $\lim_{x\to a}\frac{f(x)}{g(x)}$ might equal any number or even fail to exist! [1ex] *Condensed: The "form" $\frac{0}{0}$ is indeterminate.*

1.2 Other Indeterminate Forms

Indeterminate Forms
Indeterminate Forms

- The most basic indeterminate form is $\frac{0}{0}$.
- It is *indeterminate* because, if $\lim_{x\to a}f(x)=\lim_{x\to a}g(x)=0$, then $\lim_{x\to a}\frac{f(x)}{g(x)}$ might equal any number or even fail to exist!
- Specific cases: $\lim_{x\to 0}\frac{\sin x}{x}$, $\lim_{x\to\infty}\frac{e^{-x}}{\sin(1/x)}$, $\cdots$.
- The forms $\frac{\infty}{\infty}$ and $0\cdot\infty$ are also *indeterminate.*

- They are actually equivalent to $\frac{0}{0}$, since any specific case of one can be recast as a specific case of another.
- For example,

$$\begin{array}{ccccc} \lim_{x\to 0^+} x\ln x & = & \lim_{x\to 0^+} \dfrac{\ln x}{1/x} & = & \lim_{x\to 0^+} \dfrac{x}{1/\ln x} \\ 0\cdot\infty & & \dfrac{\infty}{\infty} & & \dfrac{0}{0} \end{array}$$

Quiz

Quiz

1. Give the limit for $\left\{1 - \dfrac{n}{n+2}\right\}_{n=1}^{\infty}$

   (a) 0, (b) 1, (c) 2.

2. Give the limit for $\left\{1 + \dfrac{n}{n+2}\right\}_{n=1}^{\infty}$

   (a) 0, (b) 1, (c) 2.

2 L'Hôpital's Rule

2.1 L'Hôpital's Rule

L'Hôpital's Rule

L'Hôpital's Rule

Suppose that $\lim_{x\to\star} \dfrac{f(x)}{g(x)}$ is a $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$. Then $\lim_{x\to\star} \dfrac{f(x)}{g(x)} = \lim_{x\to\star} \dfrac{f'(x)}{g'(x)}$

Examples 2.

$$\lim_{x\to 2} \frac{x-2}{x^2-4} \quad \left(\frac{0}{0}\right) \quad = \lim_{x\to 2} \frac{1}{2x} = \frac{1}{2\cdot 2} = \frac{1}{4}$$

$$\lim_{x\to 0} \frac{\sin x}{x} \quad \left(\frac{0}{0}\right) \quad = \lim_{x\to 0} \frac{\cos x}{1} = \frac{1}{1} = 1$$

$$\begin{aligned} \lim_{x\to 0} \frac{x-\sin x}{x\sin x} \quad \left(\frac{0}{0}\right) \quad &= \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} \quad \left(\frac{0}{0}\right) \text{ still} \\ &= \lim_{x\to 0} \frac{\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0 \end{aligned}$$

What is Wrong with This?
What is *wrong* with the following argument?

$$\lim_{x\to 0^+} \frac{x^2}{\sin x} \left(\frac{0}{0}\right) = \lim_{x\to 0^+} \frac{2x}{\cos x} = \lim_{x\to 0^+} \frac{2}{-\sin x} = \frac{2}{-0} = -\infty$$

Remark
L'Hôpital's rule *does not apply* in cases where the numerator or the denominator has a finite non-zero limit!!! For example,

$$\lim_{x\to 0^+} \frac{2x}{\cos x} = \frac{0}{1} = 0,$$

but a blind application of L'Hôpital's rule leads *incorrectly* to

$$\lim_{x\to 0^+} \frac{2x}{\cos x} = \lim_{x\to 0^+} \frac{2}{-\sin x} = \frac{2}{-0} = -\infty$$

Quiz

Quiz

3. Give the limit for $\left\{n^{\frac{1}{n}}\right\}_{n=1}^{\infty}$

   (a) 1, (b) 2, (c) e.

4. Give the limit for $\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n=1}^{\infty}$

   (a) 1, (b) 2, (c) e.

3 More Examples

3.1 Form $\frac{0}{0}$

Form $\frac{0}{0}$ *Examples* 3.

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{\cos x - 1}{3x^2} \left(\frac{0}{0}\right) \textit{ still!}$$

$$= \lim_{x\to 0} \frac{-\sin x}{6x} \left(\frac{0}{0}\right) \textit{ still!} = \lim_{x\to 0} \frac{-\cos x}{6} = \frac{-1}{6}$$

$$\lim_{x\to 0} \frac{2\cos x - 2 + x^2}{x^4} \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{-2\sin x + 2x}{4x^3} \left(\frac{0}{0}\right) \textit{ still!}$$

$$= \lim_{x\to 0} \frac{-2\cos x + 2}{12x^2} \left(\frac{0}{0}\right) \textit{ still!}$$

$$= \lim_{x\to 0} \frac{2\sin x}{24x} \left(\frac{0}{0}\right) \textit{ still!} = \lim_{x\to 0} \frac{2\cos x}{24} = \frac{2}{24} = \frac{1}{12}$$

3.2 Form $\frac{\infty}{\infty}$

Form $\frac{\infty}{\infty}$ *Examples* 4.

$$\lim_{x\to\infty} \frac{x^2}{e^x} \ \left(\frac{\infty}{\infty}\right) \ = \lim_{x\to\infty} \frac{2x}{e^x} \ \left(\frac{\infty}{\infty}\right) \ \textit{still!}$$

$$= \lim_{x\to\infty} \frac{2}{e^x} \ = \frac{2}{\infty} = 0$$

$$\lim_{x\to\infty} \frac{\ln(3^x + 2^x)}{x} \ \left(\frac{\infty}{\infty}\right) \ = \lim_{x\to\infty} \frac{3^x \ln 2 + 2^x \ln 2}{3^x + 2^x}$$

$$= \lim_{x\to\infty} \frac{\ln 3 + (2/3)^x \ln 2}{1 + (2/3)^x} = \frac{\ln 3 + 0}{1 + 0} = \ln 3$$

3.3 Limit Properties of $\ln x$ and e^x

Limit Properties of $\ln x$

$$\lim_{x\to 0^+} x^\alpha \ln x = 0, \quad \alpha > 0.$$

Proof.

$$\lim_{x\to 0^+} x^\alpha \ln x \ \ (0 \cdot \infty) \ = \lim_{x\to 0^+} \frac{\ln x}{x^{-\alpha}} \ \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0^+} \frac{1/x}{-\alpha x^{-\alpha-1}} = \lim_{x\to 0^+} \frac{x^\alpha}{-\alpha} = 0$$

$$\lim_{x\to\infty} \frac{\ln x}{x^\alpha} = 0, \quad \alpha > 0.$$

Proof.

$$\lim_{x\to\infty} \frac{\ln x}{x^\alpha} \ \left(\frac{\infty}{\infty}\right) \ = \lim_{x\to\infty} \frac{1/x}{\alpha x^{\alpha-1}} = \lim_{x\to\infty} \frac{1}{\alpha x^{\alpha-1}} = 0$$

Remark

- $\ln x$ tends to infinity *slower than* any positive power of x.
- Any positive power of $\ln x$ tends to infinity *slower than* x.

Limit Properties of e^x

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

Proof.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^n}{e^x} \ \left(\frac{\infty}{\infty}\right) \ &= \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} \ \left(\frac{\infty}{\infty}\right) \\
&= \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} \ \left(\frac{\infty}{\infty}\right) \\
&= \cdots \\
&= \lim_{x\to\infty} \frac{n!}{e^x} = \frac{n!}{\infty} = 0
\end{aligned}$$

Remark
e^x tends to infinity *fasterr than* any positive power of x.

3.4 Exponential Forms: 1^∞, 0^0, and ∞^0

Exponential Forms: 1^∞, 0^0, and ∞^0
Since $\ln x^p = p \ln x$, the log can be used to convert each of exponential forms 1^∞, 0^0, and ∞^0 to $0 \cdot \infty$:

$$\ln(1^\infty) = \infty \cdot 0, \quad \ln(0^0) = \infty \cdot 0, \quad \ln(\infty^0) = 0 \cdot \infty.$$

If $\lim\limits_{x\to\star} \ln f(x) = L$, then $\lim\limits_{x\to\star} f(x) = e^L$.
Example 5.

$$\lim_{x\to 0^+} x^x \ \left(0^0\right) \ = e^0 = 1$$

$$\begin{aligned}
\lim_{x\to 0^+} \ln(x^x) \ \left(\ln 0^0\right) \ &= \lim_{x\to 0^+} x \ln x \ \ (0 \cdot \infty) \\
&= \lim_{x\to 0^+} \frac{\ln x}{1/x} \ \left(\frac{\infty}{\infty}\right) \ = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0
\end{aligned}$$

Example 6.
More Exponential Forms: 1^∞, 0^0, and ∞^0

$$\lim_{x\to\infty} x^{1/x} \ \left(\infty^0\right) \ = e^0 = 1$$

$$\lim_{x\to 0^+} (1+x)^{1/x} \ \left(1^\infty\right) \ = e^1 = e$$

$$\lim_{x\to\infty} \ln x^{1/x} \;\left(\ln \infty^0\right) = \lim_{x\to\infty} \frac{\ln x}{x} \;\left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

$$\lim_{x\to 0^+} \ln(1+x)^{1/x} \;(\ln 1^{\infty}) = \lim_{x\to 0^+} \frac{\ln(1+x)}{x} \;\left(\frac{0}{0}\right)$$
$$= \lim_{x\to 0^+} \frac{1}{1+x} = 1$$

- Set $x = n$, we have the familiar result: as $n \to \infty$, $n^{\frac{1}{n}} \to 1$.
- Set $x = 1/n$, we have: as $n \to \infty$, $\left(1 + \frac{1}{n}\right)^n \to e$.

3.5 Form $\infty - \infty$

Form $\infty - \infty$
The form $\infty - \infty$ can be handled by converting it to a quotient.

$$\lim_{x\to(\pi/2)^-} (\tan x - \sec x) \;(\infty - \infty) = \lim_{x\to(\pi/2)^-} \frac{\sin x - 1}{\cos x} \;\left(\frac{0}{0}\right)$$
$$= \lim_{x\to(\pi/2)^-} \frac{\cos x}{-\sin x} = \frac{0}{-1} = 0$$

$$\lim_{x\to\infty} \left(e^{x+e^{-x}} - e^x\right) \;(\infty - \infty) = \lim_{x\to\infty} \frac{e^{e^{-x}} - 1}{e^{-x}} \;\left(\frac{0}{0}\right)$$
$$= \lim_{x\to\infty} \frac{e^{e^{-x}}(-e^{-x})}{-e^{-x}} = \lim_{x\to\infty} e^{e^{-x}} = e^0 = 1$$

$$\tan x - \sec x = \frac{\sin x}{\cos x} - \frac{1}{\cos x} = \frac{\sin x - 1}{\cos x}$$
$$e^{x+e^{-x}} - e^x = e^x\left(e^{e^{-x}} - 1\right) = \frac{e^{e^{-x}} - 1}{e^{-x}}$$

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