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NUMERICAL ANALYSIS

First Exam **Math 4364 (Fall 2011)** **October 13, 2011**

This exam has 3 questions, for a total of 100 points.
Please answer the questions in the spaces provided on the question sheets.
If you run out of room for an answer, continue on the back of the page.

Name and ID: ______________________________

$\boxed{35 \text{ points}}$ 1. Let

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

a) Determine the Cholesky LL^T factorization of A.

b) Use the Cholesky LL^T factorization to solve the system

$$\begin{aligned} 2x_1 - x_2 &= 3 \\ -x_1 + 2x_2 - x_3 &= -3 \\ -x_2 + 2x_3 &= 0 \end{aligned}$$

a) Refer to TEST 1 PROBLEM 3

Col 1: $m_{21} = -\frac{1}{2}$, $m_{31} = 0$
$(E_2 + \frac{1}{2}E_1) \to (E_2)$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Col 2: $m_{32} = -\frac{2}{3}$
$(E_3 + \frac{2}{3}E_2) \to (E_2)$

$$U = \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}, \quad u_{ii} > 0$$

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$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}, \quad A = LU,$$

A is s.p.d because $A^T = A$ and $u_{ii} > 0$ $\forall i$.

$$A = LDL^T, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix}, \quad D^{\frac{1}{2}} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}$$

The Cholesky factorization of A is

$$A = \bar{L}\bar{L}^T \quad \text{where} \quad \bar{L} = LD^{\frac{1}{2}} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} & 0 \\ 0 & -\frac{\sqrt{2}}{\sqrt{3}} & \frac{2}{\sqrt{3}} \end{bmatrix}$$

b) Refer to TEST1 PROBLEM 1b)

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$$Ax = b \quad \text{where} \quad b = [3, -3, 0]^T$$

$A = \bar{L}\bar{L}^T \Rightarrow$

① $\bar{L}y = b \Rightarrow y = \left[\frac{3}{\sqrt{2}}, -\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2}\right]^T$

② $\bar{L}^T x = y \Rightarrow x = \left[\frac{3}{4}, -\frac{3}{2}, -\frac{3}{4}\right]^T$

or

$A = LDL^T \Rightarrow$

① $Ly = b \Rightarrow y = \left[3, -\frac{3}{2}, -1\right]^T$

② $Dz = y \Rightarrow z = \left[\frac{3}{2}, -1, -\frac{3}{4}\right]^T$

③ $L^T x = z \Rightarrow x = \left[\frac{3}{4}, -\frac{3}{2}, -\frac{3}{4}\right]^T$

$\boxed{\text{35 points}}$ 2. We apply the Gauss-Seidel iterative method to solve the system defined in Problem 1.b.

a) Find the first two iterations of the Gauss-Seidel method using $x^0 = (0,0,0)^T$.

b) Let x denote the actual solution (computed in Problem 1.b) and $\tilde{x}$ denote approximate solution (computed in Problem 2.a). Compute the relative error and its bound

$$\frac{\|x-\tilde{x}\|_\infty}{\|x\|_\infty} \quad \text{and} \quad K_\infty(A)\frac{\|b-A\tilde{x}\|_\infty}{\|b\|_\infty}$$

a) Refer to TEST 2 PROBLEM 1.b)

The G. S. iterations are, for $k = 1, 2, \ldots$

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$$x_1^{(k)} = \frac{1}{2}x_2^{(k-1)} + \frac{3}{2}$$

$$x_2^{(k)} = \frac{1}{2}x_1^{(k)} + \frac{1}{2}x_3^{(k-1)} - \frac{3}{2}$$

$$x_3^{(k)} = \frac{1}{2}x_2^{(k)}$$

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For $k=1$, from $x^{(0)} = (0,0,0)^T$, we have

5 $\quad x^{(1)} = \left(\frac{3}{2}, -\frac{3}{4}, -\frac{3}{8}\right)^T$

For $k=2$, we have

5 $\quad x^{(2)} = \left(\frac{9}{8}, -\frac{9}{8}, -\frac{9}{16}\right)^T.$

b) Refer to TEST2 PROBLEM 2.

From 1b), $x = \left[\frac{3}{4}, -\frac{3}{2}, -\frac{3}{4}\right]^T$, $\|x\|_\infty = \max\{|x_i|\} = \frac{3}{2}$

From 2a), $\tilde{x} = \left(\frac{9}{8}, -\frac{9}{8}, -\frac{9}{16}\right)^T$,

the error $e = x - \tilde{x} = \left(-\frac{3}{8}, -\frac{3}{8}, -\frac{3}{16}\right)^T$,

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$\|e\|_\infty = \|x - \hat{x}\|_\infty = \frac{3}{8}$

The relative error is

$$\frac{\|x - \tilde{x}\|_\infty}{\|x\|_\infty} = \frac{1}{4}$$

The residual $r = b - A\tilde{x} = \left(-\frac{3}{8}, -\frac{3}{16}, 0\right)^T$

$\|r\|_\infty = \|b - A\hat{x}\|_\infty = \frac{3}{8}$, $\|b\|_\infty = 3$

The matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, $\|A\|_\infty = 4$

From 1.a), $A = LDL^T$, then

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$A^{-1} = L^{-T} D^{-1} L^{-1}$ where $L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix}$

$D^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{3}{4} \end{bmatrix}$

$= \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$, $\|A^{-1}\|_\infty = 2$

$k_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = 8$

The error bound is

$$k_\infty(A) \frac{\|b - A\hat{x}\|_\infty}{\|b\|_\infty} = 8 \frac{3/8}{3} = 1$$

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$\boxed{30 \text{ points}}$ 3. We apply the (Symmetric) Power method to approximate the most dominant eigenvalue of the matrix A defined in Problem 1.a. Find the first three iterations of the Power method using $x^0 = (1, 0, 0)^T$.

Refer to TEST 3 PROBLEM 3.

The (symmetric) power method is

2/ $x^{(0)} = x^{(0)} / \|x^{(0)}\|_2$

For $k = 1, 2, \ldots$

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$$y = A x^{(k-1)}$$
$$\mu^{(k)} = y^T x^{(k-1)}$$
$$x^{(k)} = y / \|y\|_2$$

$x^{(0)} = (1,0,0)^T$, $\|x^{(0)}\|_2 = 1$

$k=1$

$x^{(0)} = x^{(0)} / \|x^{(0)}\|_2 = (1, 0, 0)^T$

4 $y = A x^{(0)} = (2, -1, 0)^T$, $\|y\|_2 = \sqrt{5}$

$\mu^{(1)} = y^T x^{(0)} = 2$

4/ $x^{(1)} = \dfrac{y}{\|y\|_2} = \dfrac{1}{\sqrt{5}}(2, -1, 0)^T$

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.

$k=2$

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$$y = A\,x^{(1)} = \frac{1}{\sqrt{5}}(5,-4,1)^T; \qquad \|y\|_2 = \left(42/5\right)^{\frac{1}{2}} = \sqrt{42}/\sqrt{5}$$

$$\mu^{(2)} = y^T x^{(1)} = \frac{14}{5}$$

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$$x^{(2)} = \frac{y}{\|y\|_2} = \frac{1}{\sqrt{42}}(5,-4,1)^T$$

$k=3$

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$$y = A\,x^{(2)} = \frac{1}{\sqrt{42}}(14,-14,6), \quad \|y\|_2 = \frac{2\sqrt{107}}{\sqrt{42}}$$

$$\mu^{(3)} = y^T x^{(2)} = 22/7$$

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$$x^{(3)} = \frac{y}{\|y\|_2} = \frac{1}{2\sqrt{107}}(14,-14,6)$$
