NUMERICAL ANALYSIS

Exam II **Math 4364 (Fall 2011)** **December 1, 2011**

This exam has 4 questions, for a total of 120 points.
Please answer the questions in the spaces provided on the question sheets.
If you run out of room for an answer, continue on the back of the page.

Name and ID: Solution Keys

35 points 1. In some early computers, which lacked division in hardware, the division $1/c$ of any given floating-point number c ($\neq 0$) was computed by the iteration

$$x_{n+1} = x_n(2 - cx_n), \quad n = 0, 1, 2, \ldots. \tag{1}$$

using only multiplication and addition.

10/ a) Show that, if the sequence x_n generated by (1) converges to $1/c$, then

$$\lim_{n\to\infty} \frac{|x_{n+1} - 1/c|}{|x_n - 1/c|^2} = c$$

10/ b) The division $1/c$ satisfies the equation

$$f(x) = 0 \quad \text{with} \quad f(x) = x^{-1} - c.$$

Show that Newton's method for this equation gives the iteration (1).

10/ c) For $c = 3$ and $x_0 = 0.3$, use Newton's method to find first two iterates x_1 and x_2.

5/ d) Given that $1/3 \approx \mathbf{0.3333333333333333}\cdots$. What can you say, based on your calculation in c), about the number of the correct digits for the third and fourth iterates x_3 and x_4?

1. a) Iteration (1)

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$$x_{n+1} = x_n(2 - c x_n)$$

$$\Leftrightarrow \quad x_{n+1} = x_n + x_n(1 - c x_n)$$

$$\Leftrightarrow \quad -x_{n+1} = -x_n - x_n(1 - c x_n)$$

$$\Leftrightarrow \quad -c x_{n+1} = -c x_n - c x_n(1 - c x_n)$$

$$\Leftrightarrow \quad 1 - c x_{n+1} = (1 - c x_n) - c x_n(1 - c x_n)$$

$$\Leftrightarrow \quad 1 - c x_{n+1} = (1 - c x_n)^2$$

$$\Leftrightarrow \quad \frac{1}{c} - x_{n+1} = c\left(\frac{1}{c} - x_n\right)^2$$

$$\Leftrightarrow \quad \frac{\frac{1}{c} - x_{n+1}}{\left(\frac{1}{c} - x_n\right)^2} = c$$

Thus
$$\lim_{n \to \infty} \frac{|x_{n+1} - 1/c|}{|x_n - 1/c|^2} = c$$

i.e, $x_n \to 1/c$ quadratically

Alternative Solution

Iteration (1)

$$(*) \quad x_{n+1} = g(x_n) \qquad \text{where } g(x) = x(2 - cx)$$

We have $g'(x) = 2 - 2cx$, $g''(x) = -2c$

Thus $g'(1/c) = 0$, $g''(\frac{1}{c}) = -2c$

Also we have $1/c = g(1/c)$ $(**)$

From $(*)$ and $(**)$,

$$x_{n+1} - 1/c = g(x_n) - g(1/c) = \overset{0}{g'(\tfrac{1}{c})}(x_n - \tfrac{1}{c}) + \tfrac{1}{2} g''(\xi_n)(x_n - \tfrac{1}{c})^2$$

with ξ_n between x_n and $1/c$.

Then
$$\frac{x_{n+1}-\frac{1}{c}}{(x_n-1/c)^2} = \frac{1}{2} g''(\xi_n) = -c$$

Thus,
$$\lim_{n\to\infty} \frac{|x_{n+1}-1/c|}{|x_n-\frac{1}{c}|^2} = c$$

i.e. $x_n \to 1/c$ quadratically.

1.b) 10/

Newton's method for equation $f(x) = x^{-1} - c = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ $\quad \boxed{f'(x) = -x^{-2}}$

$$= x_n - \frac{(x_n^{-1}-c)}{(-x_n^{-2})}$$

$$= x_n + x_n^2(x_n^{-1}-c)$$

$$= x_n + x_n(1-cx_n)$$

Thus
$$x_{n+1} = x_n(2-cx_n)$$

Newton's method becomes iteration (1)

1.c) 10/

$x_1 = x_0(2-3x_0) = 0.\underbrace{33}_{\text{2 digits correct}}00\cdots$, $\quad x_2 = x_1(2-3x_1)$

$x_0 = 0.3$

$= 0.\underbrace{3333}_{\text{4 digits correct}}0000$

1.d) 5/

$x_n \to 1/3 \simeq 0.333333\cdots$ quadratically implies that if x_n has t correct digits, x_{n+1} has $2t$ correct digits.

Then based on c) x_3 has 8 correct digits
x_4 has 16 correct digits.

$\boxed{\text{30 points}}$ 2. Consider the fixed point problem $x = g(x)$ where $g \in C^2[a,b]$ such that $g(x) \in [a,b]$ for all $x \in [a,b]$. Suppose that a positive constant $k < 1$ exists with $|g'(x)| \le k$ for all $x \in (a,b)$. Denote by p the unique fixed point in $[a,b]$. Prove that, if $g'(p) = 0$, then for any $p_0 \neq p$ in $[a,b]$, the sequence

$$p_n = g(p_{n-1}), \quad \text{for } n = 1, 2, \cdots,$$

converges *quadratically* to p.

Proof: (i) $g \in C^2[a,b]$, $p \in [a,b]$, then Taylor expansion gives

5/ $\forall x \in [a,b]$, $\quad g(x) = g(p) + g'(p)(x-p) + \frac{1}{2} g''(\xi)(x-p)^2$

where ξ lies between x and p.

(ii) Since $g(p) = p$ and $g'(p) = 0$,

5/ (*) $\quad g(x) = p + \frac{1}{2} g''(\xi)(x-p)^2$

(iii) Since $g(x) \in [a,b]$ for all $x \in [a,b]$,

10/ thus $p_n \in [a,b]$, $\forall n \geq 1$, and, set $x = p_n$ in (*),

$$p_{n+1} = g(p_n) = p + \frac{1}{2} g''(\xi_n)(p_n - p)^2$$

where ξ_n lies between p_n and p.

Therefore
$$p_{n+1} - p = \frac{1}{2} g''(\xi_n)(p_n - p)^2$$

(iv) Since $|g'(x)| \le k < 1$ for all $x \in [a,b]$, it follows

10/ from the Fixed-point Theorem that $p_n \to p$.

As ξ_n is between p_n and p, so $\xi_n \to p$, and

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \frac{1}{2}\left|g''(p)\right|.$$

i.e. $p_n \to p$ quadratically.

$\boxed{\text{35 points}}$ 3. Show that the sequence $x_n = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}$ generated by the iteration

$$x_{n+1} = 1 + \frac{1}{x_n}, \quad n = 0, 1, 2, \cdots$$

with $x_0 = 1$ converges linearly to the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$.

Proof. * The sequence $\{x_n\}$ is generated by the fixed-point iteration

5/

$$x_{n+1} = g(x_n), \quad n = 0, 1, 2, \ldots$$

where $g(x) = 1 + \frac{1}{x}$

* The golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ is a fixed-point of g.

5/ i.e.

$$\varphi = g(\varphi) \Longleftrightarrow \varphi = 1 + \frac{1}{\varphi} \Longleftrightarrow \frac{1+\sqrt{5}}{2} = 1 + \frac{1}{\frac{1+\sqrt{5}}{2}}$$

$$RHS = 1 + \frac{2}{1+\sqrt{5}} = 1 + \frac{2(\sqrt{5}-1)}{4} = 1 + \frac{\sqrt{5}-1}{2} = \frac{1+\sqrt{5}}{2} = LHS$$

* We have ① $g \in C^1[\overset{a}{1}, \overset{b}{2}]$

② $g(x) \in [1,2], \ \forall x \in [1,2],$ ~~we~~

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$\Updownarrow$ $\forall x \in [1,2], \Longleftrightarrow 1 \leq x \leq 2 \Longleftrightarrow 1 \geq \frac{1}{x} \geq \frac{1}{2} \Longleftrightarrow 2 \geq 1 + \frac{1}{2} \geq \frac{3}{2}$

$\Rightarrow g(x) \in [1,2]$

③ $|g'(x)| \leq k < 1, \ \forall x \in [1,2]$

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$\Updownarrow$ $\forall x \in [1,2] \Longleftrightarrow 1 \leq x \leq 2 \Longleftrightarrow 1 \geq \frac{1}{x} \geq \frac{1}{2} \Longleftrightarrow 1 \geq \frac{1}{x^2} \geq \frac{1}{4}$

$\Rightarrow |g'(x)| = \frac{1}{x^2} \leq k < 1, \ \forall x \in (1,2)$

5/ ④ $\varphi \in [1,2]$ and $g'(\varphi) \neq 0$

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It follows from the <u>Fixed-Point Theorem</u> that the sequence x_n converges linearly to φ.

$\boxed{\text{20 points}}$ 4. Let $x, y \in R^n$ and A be an $n \times n$ matrix. Show that

a) $\det(A + xy^t) = (1 + y^t A^{-1} x)\det(A)$;

b) if A^{-1} exists and $1 + y^t A^{-1} x \neq 0$, then $(A + xy^t)^{-1} = A^{-1} - \frac{A^{-1}xy^tA^{-1}}{1+y^tA^{-1}x}$.

Hint: Use the following results in Test 5: Let $u, v \in R^n$ and I be the identify matrix. Then we have

1) $\det(I + uv^t) = 1 + v^t u$;

2) if $1 + v^t u \neq 0$, then $(I + uv^t)^{-1} = I - \frac{uv^t}{1+v^tu}$

Proof a)

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$$\det(A + xy^t) = \det[A(I + \underbrace{A^{-1}x}_{\overset{\shortparallel}{u}}\ \underbrace{y^t}_{\overset{\shortparallel}{v}})]$$

$$= \det[A(I + uv^t)] = \det(A)\det(I + uv^t)$$

Result 1) ⇒

$$= (1 + v^t u)\det(A) = (1 + y^t A^{-1} x)\det(A)$$

$u = A^{-1}x$
$v = y$

b) if A^{-1} exists and $1 + y^t A^{-1} x \neq 0$,
$\Rightarrow \det(A) \neq 0$

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then it follows from a) that $\det(A + xy^t) \neq 0$,

Thus $(A + xy)^{-1}$ exists.

$$(A + xy^t)^{-1} = [A(I + \underbrace{A^{-1}x}_{\overset{\shortparallel}{u}}\ \underbrace{y^t}_{\overset{\shortparallel}{v}})]^{-1} = (I + uv^t)^{-1}A^{-1}$$

Result 2) ⇒

$$= \left(I - \frac{uv^t}{1+v^tu}\right)A^{-1} = A^{-1} - \frac{uv^tA^{-1}}{1+v^tu}$$

$u = A^{-1}x$
$v = y$

$$= A^{-1} - \frac{A^{-1}xy^tA^{-1}}{1 + y^tA^{-1}x}.$$

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.