NUMERICAL ANALYSIS

First Test — Math 4364 (Fall 2011) — September 6, 2011

This exam has 3 questions, for a total of 100 points.
Please answer the questions in the spaces provided on the question sheets.
If you run out of room for an answer, continue on the back of the page.

Name and ID: Solution Keys

[30 points] 1. Let

$$A = \begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

a) Determine the $LU$ factorization of the form $PA = LU$ for the matrix $A$.

b) Use the factorization to solve the system

$$\begin{aligned} 2x_2 + 3x_3 &= 5 \\ x_1 + x_2 - x_3 &= 1 \\ -x_2 + x_3 &= 0 \end{aligned}$$

Col 1:

a) $(E_1) \leftrightarrow (E_2)$ : $m_{21} = 0$, $m_{31} = 0$

Col 2: $m_{32} = -\frac{1}{2}$

$(E_3 + \frac{1}{2} E_2) \to (E_3)$

15

$U = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & \frac{5}{2} \end{pmatrix}$ 5, $L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$ 5, & $P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 5

b) $Ax = b$, $b = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$ 3

$PAx = Pb$

15

① $Ly = Pb = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$ : 6

$y_1 = 1$

$y_2 = 5$

$-\frac{1}{2} y_2 + y_3 = 0 \Rightarrow y_3 = \frac{5}{2}$

② $Ux = y$ : $\frac{5}{2} x_3 = y_3 \Rightarrow x_3 = 1$

$2x_2 + 3x_3 = 5 \Rightarrow x_2 = 1$

6

$x_1 + x_2 - x_3 = 1 \Rightarrow x_1 = 1$

$\boxed{\text{40 points}}$ 2. Let $A$ be an $n \times n$ nonsingular matrix and $b$ be an $n$ column vector.

a) Determine the number of operations needed to perform the $LU$ factorization of the form $PA = LU$ for the matrix $A$.

b) Use the factorization to solve the system $Ax = b$ and determine the number of operations needed.

c) Use the factorization to compute $\det A$ and determine the number of operations needed.

a) For $i = 1, \dots, n-1$

$$\begin{bmatrix} x & x & x & x \\ & a_{ii} & x & x \\ & a_{pi} & x & x \\ & x & x & x \end{bmatrix} \xrightarrow{(E_i)\leftrightarrow(E_p)} \begin{bmatrix} x & x & x & x \\ & a_{ii} & x & x \\ & a_{ji} & x & x \\ & x & x & x \end{bmatrix}$$

$\forall j = i+1, \dots, n$

$$(E_j - m_{ji}E_i) \to (E_j) \qquad m_{ji} = \frac{a_{ji}}{a_{ii}}$$

$$\begin{bmatrix} x & x & x & x \\ & a_{ii} & x & x \\ & 0 & x & x \\ & 0 & x & x \end{bmatrix} \longrightarrow \begin{bmatrix} x & x & x \\ & x & x \\ 0 & & x \end{bmatrix} = U$$

$(n-i)^2$

$$L = \begin{pmatrix} 1 & \\ m_{ji} & 1 \end{pmatrix} \qquad P = \begin{pmatrix} e_r^T \\ e_i^T \end{pmatrix}$$

$$\text{flops} \sim \sum_{i=1}^{n-1} 2(n-i)^2 \sim 2 \cdot \frac{1}{3} n^3$$

$(1(*), 1(-))$

15

b) $Ax = b$

$\Leftrightarrow Ly = Pb \sim n^2$ $\quad 2 \cdot \frac{1}{2} n^2$ elms $(1(*), 1(-))$ $\quad 4$

$Ux = y \sim n^2$ $\quad 4$

3 3

15

c). $\det(A) = \det(P^T)\det(L)\det(U)$

$$= (-1)^k \prod_{i=1}^{n-1} u_{ii} \sim n-1$$

10

30 points 3. Let

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

15/ a) Determine the $LU$ factorization of the form $A = LU$ for the matrix $A$ and show that $A$ is positive definite. 10 5

7/ b) Use the $LU$ factorization to determine the $LDL^t$ factorization of $A$.

8/ c) Use the $LDL^t$ factorization to determine the Cholesky factorization of $A$.

a) col 1: $m_{21} = -\frac{1}{2}$, $m_{31} = 0$

$(E_2 + \frac{1}{2}E_1) \to (E_2)$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

col 2: $m_{32} = -\frac{2}{3}$

$(E_3 + \frac{2}{3}E_2) \to (E_2)$

$$U = \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}, \quad A = LU$$

$A$ is symmetric, it is pos. def because $u_{ii} > 0$, $\forall i = 1, \cdots, 3$

2) $A = LDL^t$,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \quad D = \operatorname{diag}(d_i) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

column scaling

3). $A = \bar{L}\bar{L}^t$:

$$\bar{L} = L \operatorname{diag}(\sqrt{d_i}) = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \frac{\sqrt{6}}{2} & 0 \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}$$

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.

$$= \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{2} & 0 \\ 0 & -\frac{\sqrt{6}}{3} & \frac{2}{\sqrt{3}} \end{bmatrix}.$$