NUMERICAL ANALYSIS

Second Test **Math 4364 (Fall 2011)** **September 22, 2011**

This exam has 2 questions, for a total of 100 points.
Please answer the questions in the spaces provided on the question sheets.
If you run out of room for an answer, continue on the back of the page.

Name and ID: Solution Keys

60 points 1. We apply the Jacobi, the Gauss-Seidel and the SOR iterative methods for the following linear system, using $x^0 = 0$:

The solution is $x = (1, 1, 1)$.

$$2x_1 - x_2 = 1$$
$$-x_1 + 2x_2 - x_3 = 0$$
$$-x_2 + 2x_3 = 1$$

a) Find the first two iterations of the Jacobi method.

b) Find the first two iterations of the Gauss-Seidel method.

c) Find the first two iterations of the SOR method with $\omega = 1.2$.

20// a) The Jacobi iterations are, for $k = 1, 2, \cdots$,

10//
$$\begin{cases} x_1^{(k)} = \frac{1}{2} x_2^{(k-1)} + \frac{1}{2} \\ x_2^{(k)} = \frac{1}{2} x_1^{(k-1)} + \frac{1}{2} x_3^{(k-1)} \\ x_3^{(k)} = \frac{1}{2} x_2^{(k-1)} + \frac{1}{2} \end{cases}$$

For $k=1$, from $x^{(0)} = (0, 0, 0)^T$, we have

5// $x^{(1)} = (\frac{1}{2}, 0, \frac{1}{2})^T$.

For $k=2$, from $x^{(1)} = (\frac{1}{2}, 0, \frac{1}{2})^T$, we have

5// $x^{(2)} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$

20// b) The G. S. iterations are, for $k=1, 2, \ldots$

10// $$\begin{cases} x_1^{(k)} = \frac{1}{2} x_2^{(k-1)} + \frac{1}{2} \\ x_2^{(k)} = \frac{1}{2} x_1^{(k)} + \frac{1}{2} x_3^{(k-1)} \\ x_3^{(k)} = \frac{1}{2} x_2^{(k)} + \frac{1}{2} \end{cases}$$

For $k=1$, from $x^{(0)} = (0,0,0)^T$, we have

5// $$x^{(1)} = \left(\frac{1}{2}, \frac{1}{4}, \frac{5}{8}\right)^T$$

For $k=2$, from $x^{(1)} = \left(\frac{1}{2}, \frac{1}{4}, \frac{5}{8}\right)^T$, we have

5// $$x^{(2)} = \left(\frac{5}{8}, \frac{5}{8}, \frac{13}{16}\right)^T.$$

20// c) The SOR iterations with $\omega = 1.2$, are, for $k=1,2,$

10// $$\begin{cases} x_1^{(k)} = -0.2 x_1^{(k-1)} + 0.6 x_2^{(k-1)} + 0.6 \\ x_2^{(k)} = -0.2 x_2^{(k-1)} + 0.6 x_1^{(k)} + 0.6 x_3^{(k-1)} \\ x_3^{(k)} = -0.2 x_3^{(k-1)} + 0.6 x_2^{(k)} + 0.6 \end{cases}$$

For $k=1$, from $x^{(0)} = (0,0,0)^T$, we have

5// $$x^{(1)} = (0.6, 0.36, 0.816)^T$$

For $k=2$, we have

5// $$x^{(2)} = (0.696, 0.8352, 0.9379)^T$$

10 points 2. The following linear system $Ax = b$:

$$x_1 - x_2 - x_3 = 2$$
$$x_2 - x_3 = 0$$
$$-x_3 = 1$$

has $x = (0, -1, -1)^T$ as the actual solution and $\tilde{x} = (-0.1, -1.1, -1.1)^T$ as an approximate solution. Compute the relative error and its bound

$$\frac{\|x - \tilde{x}\|_\infty}{\|x\|_\infty} \quad \text{and} \quad K_\infty(A)\frac{\|b - A\tilde{x}\|_\infty}{\|b\|_\infty}$$

5//

20// a). The error $e = x - \tilde{x} = (0.1, 0.1, 0.1)^T$.

5//

$\|e\|_\infty = \|x - \tilde{x}\|_\infty = 0.1$, $\|x\|_\infty = 1$. 5//

The relative error $\frac{\|x - \tilde{x}\|_\infty}{\|x\|_\infty} = 0.1$ 5//

20// b) The residual $r = b - A\tilde{x} = (-0.1, 0, -0.1)^T$

5//

$\|r\|_\infty = \|b - A\tilde{x}\|_\infty = 0.1$, $\|b\|_\infty = 2$.

The matrix $A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$

5// $\|A\|_\infty = 3$, $A^{-1} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$, $\|A^{-1}\|_\infty = 4$ 5//

$k_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = 12$

The error bound 5//

$$K_\infty(A) \frac{\|b - A\tilde{x}\|_\infty}{\|b\|_\infty} = 0.6$$

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.

The SOR iterations with $\omega = 1.2 = \frac{6}{5}$ are

$$x_1^{(k)} = -\frac{1}{5} x_1^{(k-1)} + \frac{6}{5}\left(\frac{1}{2} x_2^{(k-1)} + \frac{1}{2}\right)$$

$$x_2^{(k)} = -\frac{1}{5} x_2^{(k-1)} + \frac{6}{5}\left(\frac{1}{2} x_1^{(k)} + \frac{1}{2} x_3^{(k-1)}\right)$$

$$x_3^{(k)} = -\frac{1}{5} x_3^{(k-1)} + \frac{6}{5}\left(\frac{1}{2} x_2^{(k)} + \frac{1}{2}\right)$$

$$\Longleftrightarrow \quad x_1^{(k)} = -\frac{1}{5} x_1^{(k-1)} + \frac{3}{5} x_2^{k-1} + \frac{3}{5}$$

$$x_2^{(k)} = -\frac{1}{5} x_2^{(k-1)} + \frac{3}{5} x_1^{(k)} + \frac{3}{5} x_3^{(k-1)}$$

$$x_3^{(k)} = -\frac{1}{5} x_3^{(k-1)} + \frac{3}{5} x_2^{(k)} + \frac{3}{5}$$

For $k=1$, from $x^{(0)} = (0,0,0)^T$, we have

$$x^{(1)} = \left(\frac{3}{5}, \underset{\overset{\shortparallel}{3^2/5^2}}{\frac{9}{25}}, \underset{\overset{\shortparallel}{\left(27/125 + \frac{3}{5}\right)}}{\frac{3^3}{5^3} + \frac{3}{5}} = \underline{\underline{\frac{102}{125}}}\right)$$

For $k=2$

$$x^{(2)} = \left(\frac{87}{125}, \underset{\overset{\shortparallel}{\frac{522}{625}}}{\left(\frac{3}{5}\cdot\frac{174}{125}\right)}, \left(\frac{1056}{3125} + \frac{3}{5}\right)\right)$$

$$= 2931/3125$$