NUMERICAL ANALYSIS

Forth Test **Math 4364 (Fall 2011)** **November 3, 2011**

This exam has 3 questions, for a total of 100 points. Please answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Name and ID: Solution Keys

50 points

1. Heron's rule is widely used to calculate the square root of c (> 0). It is given as a fixed-point iteration

$$x_{n+1} = \frac{1}{2}(x_n + c/x_n), \quad n = 1, 2, \ldots.$$

a) Show that the sequence x_n converges quadratically to $\sqrt{c}$ by proving that

$$\lim_{n\to\infty} \frac{|x_{n+1} - \sqrt{c}|}{|x_n - \sqrt{c}|^2} = \text{constant}$$

b) The square root of c satisfies the equation

$$f(x) = 0 \quad \text{with} \quad f(x) = x^2 - c.$$

Show that Newton's method for this equation gives Heron's rule.

c) For $c = 2$ and $x_0 = 1.5$, use Newton's method to find first two iterates x_1 and x_2.

d) Given that $\sqrt{2} \approx \mathbf{1.41421356237309}5$ (with 15 correct digits shown in bold). What can you say, based on your calculation in c), about the number of the correct digits for the third and forth iterates x_3 and x_4?

1. a) Heron's rule

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$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right), \quad \forall n \geq 1$$

$$\Longleftrightarrow \quad x_{n+1} - \sqrt{c} = \frac{1}{2x_n}\left(x_n^2 - 2\sqrt{c}\, x_n + c\right)$$

$$\Longleftrightarrow \quad x_{n+1} - \sqrt{c} = \frac{1}{2x_n}\left(x_n - \sqrt{c}\right)^2$$

$$\Longleftrightarrow \quad \frac{x_{n+1} - \sqrt{c}}{(x_n - \sqrt{c})^2} = \frac{1}{2x_n}$$

Taking the limit

$$\lim_{n\to\infty} \frac{|x_{n+1} - \sqrt{c}|}{|x_n - \sqrt{c}|^2} = \lim_{n\to\infty} \frac{1}{2x_n} = \frac{1}{2\sqrt{c}} = \text{constant}$$

Thus $x_n \to \sqrt{c}$ quadratically

b) Newton's method for equation $f(x) = x^2 - c = 0$

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$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \boxed{f'(x) = 2x}$$

$$= x_n - \frac{x_n^2 - c}{2x_n}$$

Thus
$$= x_n - \frac{x_n}{2} + \frac{c}{2x_n} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$$

Newton's method becomes Heron's rule.

c) $x_1 = \frac{1}{2}\left(x_0 + \frac{2}{x_0}\right) = \underbrace{1.41}_{\text{3 digits correct}}6\cdots$, $x_2 = \frac{1}{2}\left(x_1 + \frac{2}{x_1}\right)$ ~~1.414216~~

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$$= \underbrace{1.41421}_{\text{6 correct digits}}5\cdots$$

d) $x_n \to \sqrt{2}$ quadratically implies that, if x_n has t correct digits, then x_{n+1} has $2t$ correct digits.

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Therefore, based on c), x_3 has 12 correct digits
x_4 has 24 correct digits.

1. a) Alternative Solution

Heron's rule

(1) $\quad x_{n+1} = g(x_n)$

where $g(x) = \frac{1}{2}(x + \frac{c}{x})$.

We have $g'(x) = \frac{1}{2}(1 - cx^{-2}),\ g''(x) = cx^{-3}$.

Thus, $g'(\sqrt{c}) = 0,\ g''(\sqrt{c}) = \frac{1}{\sqrt{c}}$

Also, we have

(2) $\quad \sqrt{c} = g(\sqrt{c}),$

and, from (1) − (2),

$$x_{n+1} - \sqrt{c} = g(x_n) - g(\sqrt{c})$$
$$= \underbrace{g'(\sqrt{c})}_{=0}(x_n - \sqrt{c}) + \frac{1}{2} g''(\xi_n)(x_n - \sqrt{c})^2$$

with ξ_n between x_n and c

Then, $\dfrac{x_{n+1} - \sqrt{c}}{(x_n - \sqrt{c})^2} = \frac{1}{2} g''(\xi_n)$ as $\xi_n \to \sqrt{c}$,

Taking limit,

$$\lim_{n\to\infty} \frac{|x_{n+1} - \sqrt{c}|}{|x_n - \sqrt{c}|^2} = \lim_{n\to\infty} \frac{1}{2}|g''(\xi_n)| = \frac{1}{2\sqrt{c}} = \text{constant}$$

points 2. Consider the fixed point problem $x = g(x)$ where $g \in C^1[a,b]$ such that $g(x) \in [a,b]$ for all $x \in [a,b]$. Suppose that a positive constant $k < 1$ exists with $|g'(x)| \le k$ for all $x \in (a,b)$. Denote by p the unique fixed point in $[a,b]$. Prove that, if $g'(p) \neq 0$, then for any $p_0 \neq p$ in $[a,b]$, the sequence

$$p_n = g(p_{n-1}), \qquad \text{for } n = 1, 2, \cdots,$$

converges only linearly to p.

Proof: Fixed point iteration

5/ (1) $p_{n+1} = g(p_n), \quad \forall n \ge 0$

Note that, $g \in C^1[a,b]$,

5/ $g(p_n) = g(p) + g'(\xi_n)(p_n - p)$ for some ξ_n between p and p_n.

p being the unique fixed point in $[a,b]$.

5/ $p = g(p)$

From (1), we have

$$p_{n+1} = p + g'(\xi_n)(p_n - p)$$

$$\Leftrightarrow \quad p_{n+1} - p = g'(\xi_n)(p_n - p)$$

5/ $$\Leftrightarrow \quad \frac{p_{n+1} - p}{p_n - p} = g'(\xi_n)$$

Taking the limit as $\xi_n \to p$

5/ $$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to} |g'(\xi_n)| = |g'(p)| \le k < 1$$

Therefore, $p_n \to p$ linearly

3. a) Show that the sequence $p_n = \frac{1}{n}$ converges linearly to $p = 0$.

b) Show that the sequence $p_n = 10^{-2^n}$ converges quadratically to $p = 0$.

a) $$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n - p|} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

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Thus $\{p_n\} \to p$ linearly

b) $$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n - p|^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{\left(10^{-2^n}\right)^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-2^{n+1}}} = 1$$

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Thus $p_n \to p$ quadratically

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.