NUMERICAL ANALYSIS

Test 5 **Math 4364 (Fall 2011)** **November 22, 2011**

This exam has 2 questions, for a total of 100 points.
Please answer the questions in the spaces provided on the question sheets.
If you run out of room for an answer, continue on the back of the page.

Name and ID: Solution Keys

[50 points] 1. The nonlinear system

$$x_1^2 - 4x_1 + x_2^2 = 0, \quad x_1^2 - x_2^2 - 4x_2 = 0$$

can be transformed into the fixed-point problem

$$x_1 = g_1(x_1, x_2) = \frac{x_1^2 + x_2^2}{4}, \quad x_2 = g(x_1, x_2) = \frac{x_1^2 - x_2^2}{4}$$

a) Use Theorem 10.6 (Contraction Mapping Theorem) to show that $G = (g_1, g_2)^t$ mapping $D \subset R^2$ into R^2 has a unique fixed point $p = (0,0)^t$ in

$$D = \{ (x_1, x_2)^t \mid -\frac{1}{2} \le x_1, x_2 \le \frac{1}{2} \}$$

b) Apply the fixed-point iteration with $x^{(0)} = (\frac{1}{2}, \frac{1}{3})^t$ to compute $x^{(2)}$.

c) Apply the Gauss-Seidel method with the same $x^{(0)}$ to compute $x^{(2)}$. Does the Gauss-Seidel method accelerate convergence?

a) 20

* It is easy to show that $G: D \to \mathbb{R}^2$ is continuous

* To show that $\forall x \in D$, $G(x) \in D$: we have

$$0 \le g_1(x_1, x_2) = \frac{x_1^2 + x_2^2}{4} \le \frac{1}{8}$$

$$-\frac{1}{16} \le g_2(x_1, x_2) = \frac{x_1^2 - x_2^2}{4} \le \frac{1}{16}$$

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* To show that $\left|\frac{\partial g_i(x)}{\partial x_j}\right| \le \frac{K}{2}$, $\forall i,j = 1,2$, $\forall x \in D$ for some constant $K < 1$.

we have

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$$-\frac{1}{4} \le \frac{\partial g_1(x)}{\partial x_1} = \frac{1}{2} x_1 \le \frac{1}{4}, \quad -\frac{1}{4} \le \frac{\partial g_1(x)}{\partial x_2} = \frac{1}{2} x_2 \le \frac{1}{4}$$

$$-\frac{1}{4} \le \frac{\partial g_2(x)}{\partial x_2} = \frac{1}{2} x_1 \le \frac{1}{4}, \quad -\frac{1}{4} \le \frac{\partial g_2(x)}{\partial x_2} = -\frac{1}{2} x_2 \le \frac{1}{4}$$

$\Rightarrow \left|\frac{\partial g_i}{\partial x_j}(x)\right| \le \frac{1}{4} = \frac{1/2}{2}$ for $i,j = 1,2$. i.e $K = \frac{1}{2}$

All the hypothesis of Theorem 10.6 have been satisfied.

Thus, G has a unique fixed point in D.

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(b) The fixed-point iteration generates

$$X^{(k)} = G(X^{(k-1)}), \quad \forall k \geq 1$$

$$x_1^{(k)} = \frac{1}{4}\left((x_1^{(k-1)})^2 + (x_2^{(k-1)})^2\right)$$

$$x_2^{(k)} = \frac{1}{4}\left((x_1^{(k-1)})^2 - (x_2^{(k-1)})^2\right)$$

from $X^{(0)} \in D$.

By Theorem 10.6, we have $X^{(k)} \xrightarrow{k \uparrow \infty} p$ and

$$\| X^{(k)} - p \|_\infty \leq \frac{K^k}{1-K} \| X^{(1)} - X^{(0)} \|_\infty$$

$X^{(0)} = (\frac{1}{2}, \frac{1}{3})^t$, $\quad X^{(1)} = (\frac{13}{144}, \frac{5}{144})^t = (0.0903, 0.0347)^t$.

$X^{(2)} = (0.0023, 0.0017)$

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(c) The Gauss-Seidel method uses the latest estimate $x_1^{(k)}, \ldots, x_{i-1}^{(k)}$ instead $x_1^{(k-1)}, \ldots, x_{i-1}^{(k-1)}$ to compute $x_i^{(k)}$, i.e.

$$x_1^{(k)} = \frac{1}{4}\left((x_1^{(k-1)})^2 + (x_2^{(k-1)})^2\right)$$

$$x_2^{(k)} = \frac{1}{4}\left((x_1^{(k)})^2 + (x_2^{(k-1)})^2\right)$$

Therefore, we have

$$X^{(0)} = (\tfrac{1}{2}, \tfrac{1}{3})^t, \quad X^{(1)} = (0.0903, -0.0257)^t$$

$$X_1^{(2)} = (0.0022, \; -0.000164)^t$$

$\boxed{\text{50 points}}$ 2. Let $u, v \in R^n$ and I be the identify matrix. Show that

a) $\det(I + uv^t) = 1 + v^t u$;

b) if $1 + v^t u \neq 0$, then $(I + uv^t)^{-1} = I - \frac{uv^t}{1+v^t u}$.

(a) To show that the matrix $M = I + uv^t$ has

30 eigenvalues $\lambda_1 = \cdots = \lambda_{n-1} = 1$, $\lambda_n = 1 + v^t u$.

Let λ be an eigenvalue of M with eigenvector $x \neq 0$.

Then $Mx = \lambda x \iff (I + uv^t)x = \lambda x \iff x + (v^t x)u = \lambda x$

$\iff (\lambda - 1)x = (v^t x)\, u \quad (*)$

* If $\lambda = 1$, then $v^t x = 0$. i.e $x \in v^\perp$ $= \{ y \in \mathbb{R}^n : v^T y = 0 \}$.

Since $\dim\{v^\perp\} = n-1$, then

$\lambda = 1$ is an eigenvalue of M with multiplicity $n-1$.

and eigenvectors $x^{(1)}, \ldots, x^{(n-1)}$, where $v^t x^{(j)} = 0$, $j = 1, \ldots, n-1$

* If $\lambda \neq 1$, then x and u are parallel, i.e,

$\exists\, \alpha \neq 0$ s.t. $x = \alpha u$. Then

$(\lambda - 1)\alpha u = (v^t(\alpha u))\, u$

Thus $\alpha(\lambda - 1)\, u = \alpha(v^t u)\, u \iff \lambda - 1 = v^T u$

Therefore $\lambda = 1 + v^t u$

Since $\det M = \prod_{i=1}^{n} \lambda_i$, then $\det(I + uv^t) = 1 + v^t u$.

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.

If $1+v^t u \neq 0$, then $\det(I+uv^t) = 1+v^t u \neq 0$

therefore $(I+uv^t)^{-1}$ exists.

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b) We need to show $(I+uv^t)\left(I - \dfrac{uv^t}{1+v^t u}\right) = I$.

$$(I+uv^t)\left(I - \frac{uv^t}{1+v^t u}\right) = I + uv^t - \frac{uv^t}{1+v^t u} - \frac{u(v^t u)v^t}{1+v^t u}$$

$$= I + uv^t - \frac{1}{1+v^t u}(1+v^t u)\,uv^t$$

$$= I + uv^t - uv^t$$

$$= I$$

Thus $(I+uv^t)^{-1} = I - \dfrac{uv^t}{1+v^t u}$