NUMERICAL ANALYSIS

Sample Test 1 **Math 4365 (Spring 2012)** **January 31, 2012**

20 points 1. Write out the cardinal functions $L_i(x)$ appropriate to the problem of interpolating the following table, and give the Lagrange form of the interpolating polynomial:

x	$\frac{1}{3}$	$\frac{1}{4}$	1
$f(x)$	2	-1	7

Solution The cardinal functions $L_i(x)$ are

$$L_0(x) = \frac{(x-\frac{1}{4})(x-1)}{(\frac{1}{3}-\frac{1}{4})(\frac{1}{3}-1)} = -18(x-\frac{1}{4})(x-1)$$
$$L_1(x) = \frac{(x-\frac{1}{3})(x-1)}{(\frac{1}{4}-\frac{1}{3})(\frac{1}{4}-1)} = 16(x-\frac{1}{3})(x-1)$$
$$L_2(x) = \frac{(x-\frac{1}{3})(x-\frac{1}{4})}{(1-\frac{1}{3})(1-\frac{1}{4})} = 2(x-\frac{1}{3})(x-\frac{1}{4})$$

There, the interpolating polynomial in the Lagrange form is

$$p_2(x) = -36(x-\frac{1}{4})(x-1) - 16(x-\frac{1}{3})(x-1) + 14(x-\frac{1}{3})(x-\frac{1}{4})$$

20 points 2. Construct a divided-difference diagram for the function f given in the following table, and write out the Newton form of the interpolating polynomial

x	1	$\frac{3}{2}$	0	2
$f(x)$	3	$\frac{13}{4}$	3	$\frac{5}{3}$

Solution The complete diagram is

x	$f[\cdot]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$
1	3			
		$\frac{1}{2}$		
$\frac{3}{2}$	$\frac{13}{4}$		$\frac{1}{3}$	
		$\frac{1}{6}$		-2
0	3		$-\frac{5}{3}$	
		$-\frac{2}{3}$		
2	$\frac{5}{3}$			

where the first entry in column 3 is

$$f[x_0, x_1] = \frac{\frac{13}{4}-3}{\frac{3}{2}-1} = \frac{1}{2}$$

and after completion of column 3, the first entry in column 4 is

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{\frac{1}{6} - \frac{1}{2}}{0 - 1} = \frac{1}{3}$$

Thus, the Newton form of the interpolating polynomial is

$$p_3(x) = 3 + \frac{1}{2}(x - 1) + \frac{1}{3}(x - 1)(x - \frac{3}{2}) - 2(x - 1)(x - \frac{3}{2})x$$

20 points

3. Write out the cardinal functions $H_i(x)$ and $\hat{H}_i(x)$ appropriate to the problem of interpolating the following table, and give the Hermite interpolating polynomial:

x	$f(x)$	$f'(x)$
0	2	1
1	1	2

Solution First compute the Lagrange polynomials $L_i(x)$ and their derivatives:

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - 1}{0 - 1} = 1 - x, \quad L_0'(x) = -1,$$
$$L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 0}{1 - 0} = x, \quad L_1'(x) = 1$$

The cardinal functions $H_i(x)$ and $\hat{H}_i(x)$ are

$$\begin{aligned}
H_0(x) &= [1 - 2(x - x_0)L_0'(x_0)]L_0^2(x) = (1 + 2x)(1 - x)^2, \\
\hat{H}_0(x) &= (x - x_0)L_0^2(x) = x(1 - x)^2, \\
H_1(x) &= [1 - 2(x - x_1)L_1'(x_1)]L_1^2(x) = (3 - 2x)x^2, \\
\hat{H}_1(x) &= (x - x_1)L_1^2(x) = (x - 1)x^2
\end{aligned}$$

Thus, the Hermite interpolating polynomial is

$$p_3(x) = 2(1 + 2x)(1 - x)^2 + x(1 - x)^2 + (3 - 2x)x^2 + 2(x - 1)x^2$$

20 points

4. Construct a divided-difference diagram for the function f given in the following table, and give the Hermite interpolating polynomial:

x	$f(x)$	$f'(x)$
0	2	1
1	1	2

Solution The complete diagram is

z	$f[\cdot]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$
0	2			
		1		
0	2		-2	
		-1		5
1	1		3	
		2		
1	1			

where the first entry in column 3 is $f'(x_0)$, the second entry in column 3 is

$$f[z_1, z_2] = \frac{1-2}{1-0} = -1$$

and after completion of column 3, the first entry in column 4 is

$$f[z_0, z_1, z_2] = \frac{f[z_1, z_2] - f[z_0, z_1]}{z_2 - z_0} = \frac{-1-1}{1-0} = -2$$

Thus, the Newton form of the Hermite interpolating polynomial is

$$p_3(x) = 2 + x - 2x^2 + 5x^2(x-1)$$

20 points

5. Determine the parameters a, b, c, d, e, f, g, and h so that $S(x)$ is a natural cubic spline, where

$$S(x) = \begin{cases} ax^3 + bx^2 + cx + d, & x \in [-1, 0] \\ ex^3 + fx^2 + gx + h, & x \in [0, 1] \end{cases}$$

with interpolating conditions

$$S(-1) = 1, \quad S(0) = 2, \quad S(1) = -1$$

Solution From the interpolation conditions, we have

$$\begin{aligned} S(0) = 2 \quad &\Rightarrow \quad d = 2, \quad h = 2, \\ S(-1) = 1 \quad &\Rightarrow \quad -a + b - c + d = 1, \\ S(1) = -1 \quad &\Rightarrow \quad e + f + g + h = -1 \end{aligned}$$

Since

$$S'(x) = \begin{cases} 3ax^2 + 2bx + c, & x \in [-1, 0] \\ 3ex^2 + 2fx + g, & x \in [0, 1] \end{cases}$$

we have $c = g$ from the continuity condition of S' at $x = 0$, i.e.,

$$S'(0) = c = g.$$

Also, since

$$S''(x) = \begin{cases} 6ax + 2b, & x \in [-1, 0] \\ 6ex + 2f, & x \in [0, 1] \end{cases}$$

we have $b = f$ from the continuity condition of S'' at $x = 0$, i.e.,

$$S''(0) = 2b = 2f.$$

In order for S to be a natural cubic spline, we must have

$$\begin{aligned} S''(-1) = 0 \quad &\Rightarrow \quad -6a + 2b = 0 \\ S''(1) = 0 \quad &\Rightarrow \quad 6e + 2f = 0 \end{aligned}$$

From all of these equations, we obtain

$$a = -1, \quad b = -3, \quad c = -1, \quad d = 2, \quad e = 1, \quad f = -3, \quad g = -1, \quad h = 2$$