NUMERICAL ANALYSIS

Sample Test 2 **Math 4365 (Spring 2012)** **February 16, 2012**

20 points 1. Taylor's theorem can be used to show that centered-difference formula to approximate $f'(x_0)$ can be expressed with an error formula

$$f'(x_0) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right] - \frac{h^2}{6}f'''(x_0) - \frac{h^4}{120}f^{(5)}(x_0) - \cdots$$

Find approximations of order $O(h^2)$, $O(h^4)$, and $O(h^6)$ for $f'(2)$ when $h = 0.2$ and $f(x)$ is represented by the following table

x	1.8	1.9	1.95	2.05	2.1	2.2
$f(x)$	10.889	12.703	13.706	15.924	17.149	19.855

Solution Let $M = f'(2)$ and $N_1(h) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right]$. Then we have the $O(h^2)$ approximation

$$M = N_1(h) + K_1h^2 + K_2h^4 + K_3h^6 + \cdots$$

The Richardson's Extrapolation

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{1}{4^{j-1}-1}\left(N_{j-1}\left(\frac{h}{2}\right) - N_{j-1}(h)\right), \quad j = 2, 3, \ldots,$$

gives the $O(h^{2j})$ approximation for $M = f'(2)$. From the $O(h^2)$ approximation

$$N_1(0.2) = \frac{1}{0.4}\left[f(2.2) - f(1.8)\right] = 2.5(19.855 - 10.889) = 22.414$$
$$N_1(0.1) = \frac{1}{0.2}\left[f(2.1) - f(1.9)\right] = 5(17.149 - 12.703) = 22.228$$
$$N_1(0.05) = \frac{1}{0.1}\left[f(2.05) - f(1.95)\right] = 10(15.924 - 13.706) = 22.183$$

we can compute the $O(h^4)$ approximation

$$N_2(0.2) = N_1(0.1) + \frac{1}{3}\left[N_1(0.1) - N_1(0.2)\right] = 22.167$$
$$N_2(0.1) = N_1(0.05) + \frac{1}{3}\left[N_1(0.05) - N_1(0.1)\right] = 22.183$$

and finally the $O(h^6)$ approximation

$$N_3(0.2) = N_2(0.1) + \frac{1}{15}\left[N_2(0.1) - N_2(0.2)\right] = 22.167$$

2. (a) Use Simpson's rule to approximate

$$\int_1^{1.5} x^2 \ln x dx$$

(b) Find a bound for the error in the Simpson's rule approximation in part (a).

20 points

Solution (a) Simpson's rule gives

$$\int_1^{1.5} x^2 \ln x dx \approx \frac{0.25}{3}\left[1^2\ln(1) + 4(1.25)^2\ln(1.25) + (1.5)^2\ln(1.5)\right] \approx 0.19225$$

(b) Since

$$f'(x) = 2x\ln x + x, \quad f''(x) = 2\ln x + 3, \quad f'''(x) = 2/x, \quad f^{(4)}(x) = -2/x^2,$$

the error is bounded on $[1, 1.5]$ by

$$\left|\frac{h^5}{90} f^{(4)}(x)\right| \le \frac{(0.25)^5}{90}\frac{2}{1^2} \approx 2.17\times 10^{-5}$$

20 points

3. A car laps a race track in 60 seconds. The speed of the car at each 6-second interval is determined by using a radar gun and is given from the beginning of the lap, in feet/second, by the entries in the following table. Use the Composite Simpson's rule to determine the length of the track.

Time	0	6	12	18	24	30	36	42	48	54	60
Speed	124	134	148	156	147	133	121	109	99	85	78

Solution Since the velocity $v(t)$ is the derivative of the distance $s(t)$, the total distance traveled in the 60 second interval is

$$s(60) = \int_0^{60} v(t)dt.$$

However, we do not have an explicit representation for the velocity, only its values at each 6-second interval. We can approximate the distance by doing numerical integration on the velocity. Using the Composite Simpson's rule gives the approximate length of the track as

$$\begin{aligned} s(60) &= \int_0^{60} v(t)dt \\ &\approx \frac{6}{3}(124 + 4(134+156+133+109+85) + 2(148+147+121+99) + 78) \\ &= 2 * 3700 = 7400 \text{ feet} \end{aligned}$$

20 points

4. Determine constants a, b, c and d that will produce a quadrature formula

$$\int_{-1}^{1} f(x)dx = af(-1) + bf(1) + cf'(-1) + df'(1)$$

that has degree of precision 3.

Solution The formula must be exact for $f(x) = 1$, x, x^2 and x^3. Thus

$$\begin{aligned}
f(x) = 1, \quad f'(x) = 0 \quad &\Rightarrow \quad \int_{-1}^{1} dx = 2 = a + b; \\
f(x) = x, \quad f'(x) = 1 \quad &\Rightarrow \quad \int_{-1}^{1} x dx = 0 = -a + b + c + d; \\
f(x) = x^2, \quad f'(x) = 2x \quad &\Rightarrow \quad \int_{-1}^{1} x^2 dx = 2/3 = a + b - 2c + 2d; \\
f(x) = x^3, \quad f'(x) = 3x^2 \quad &\Rightarrow \quad \int_{-1}^{1} x^3 dx = 0 = -a + b + 3c + 3d;
\end{aligned}$$

Solving the four equations gives $a = 1$, $b = 1$, $c = 1/3$, and $d = -1/3$.

20 points

5. Use the Composite Simpson's rule for $n = 4$ to approximate the value of the improper integral

$$\int_0^1 x^{-1/4} \sin x dx$$

Solution For the sin function, the 4th Taylor polynomial about zero is

$$p_4(x) = x - \frac{x^3}{6}$$

and

$$\int_0^1 x^{-1/4} p_4(x) dx = \int_0^1 \left(x^{3/4} - \frac{x^{11/4}}{6} \right) dx = \left[\frac{4}{7} x^{7/4} - \frac{2}{45} x^{15/4} \right]_0^1 = \frac{4}{7} - \frac{2}{45}$$

Now define

$$G(x) = \begin{cases} x^{-1/4}(\sin x - p_4(x)), & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

and apply the Composite Simpson's rule with $h = 1/4 = 0.25$ to $G(x)$. This gives

$$\begin{aligned}
\int_0^1 G(x) dx &= \frac{0.25}{3} \left[G(0) + 4G(0.25) + 2G(0.5) + 4G(0.75) + G(1) \right] \\
&\approx \frac{0.25}{3} \left[0 + 4(0.0000115) + 2(0.000308) + 4(0.00210) + 0.00814 \right] \approx 0.00143
\end{aligned}$$

Hence

$$\int_0^1 x^{-1/4} \sin x dx = \int_0^1 x^{-1/4} p_4(x) dx + \int_0^1 G(x) dx \approx \left(\frac{4}{7} - \frac{2}{45} \right) + 0.00143 \approx 0.5284$$