NUMERICAL ANALYSIS

Sample Test 3 **Math 4365 (Spring 2012)** **March 6, 2012**

40 points 1. Consider the initial value problem

$$y' = -2y + te^{3t}, \quad 0 \le t \le 1, \quad y(0) = 0. \tag{1}$$

(a) Use Euler's method with $h = 0.5$ to approximate the solution to equation (1).
Solution: Let $f(t, y) = -2y + te^{3t}$. use Euler's method

$$\begin{aligned}
&t_0 = 0, \quad w_0 = y(t_0) = 0,\\
&\ \text{for } i = 0, N,\\
&\quad t_{i+1} = t_i + h, \quad w_{i+1} = w_i + hf(t_i, w_i) = w_i + h(-2w_i + t_i e^{3t_i}).\\
&t_1 = t_0 + h = 0.5, \quad y(t_1) \approx w_1 = w_0 + h(-2w_0 + t_0 e^{3t_0}) = 0\\
&t_2 = t_1 + h = 1, \quad y(t_2) \approx w_2 = w_1 + h(-2w_1 + t_1 e^{3t_1}) = 1.1204223
\end{aligned}$$

(b) The exact soltuion to the initial value problem (1) is

$$y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$$

Determine an error bound for the approximation obtained in (a).
Solution: Let L be the Lipshitz constant L of $f(t, y) = -2y + te^{3t}$ and M be the maximum value of $|y''(t)|$ on $[0, 1]$. We have

$$\begin{aligned}
y'(t) &= \frac{3}{5}te^{3t} + \frac{2}{25}e^{3t} - \frac{2}{25}e^{-2t},\\
y''(t) &= \frac{9}{5}te^{3t} + \frac{21}{25}e^{3t} + \frac{4}{25}e^{-2t}\\
&\Rightarrow \quad M = \max_{t\in[0,1]} |y''(t)| = y''(1) = 53.047\\
\frac{\partial f}{\partial y}(t, y) &= -2 \quad \Rightarrow \quad L = \max_{t,y} \left|\frac{\partial f}{\partial y}(t, y)\right| = 2
\end{aligned}$$

Hence, an error bound is given by

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left(e^{L(t_i - t_0)} - 1\right)$$

At $t_1 = 0.5$, this bound is 11.3938, and at $t_2 = 1$, it is 42.3654.

(c) Use Taylor's method of order two with $h = 0.5$ to approximate the solution to equation (1)
Solution:

$$f'(t, y) = \frac{d}{dt}(-2y + te^{3t}) = -2y' + e^{3t} + 3te^{3t} = 4y + e^{3t}(1 + t)$$

Use Taylor's method of order two

$$
\begin{aligned}
& t_0 = 0, \quad w_0 = y(t_0) = 0, \\
& \text{for } i = 0, N, \\
& \quad t_{i+1} = t_i + h, \quad w_{i+1} = w_i + hf(t_i, w_i) + \frac{h^2}{2} f'(t_i, w_i) \\
& \qquad\qquad\qquad\qquad = w_i + h(-2w_i + t_i e^{3t_i}) + \frac{h^2}{2}(4w_i + e^{3t_i}(1 + t_i)) \\
& t_1 = t_0 + h = 0.5, \quad y(t_1) \approx w_1 = 0.2578125 \\
& t_2 = t_1 + h = 1, \quad y(t_2) \approx w_2 = 3.05529474
\end{aligned}
$$

(d) Use the modified Euler method with $h = 0.5$ to approximate the solution to equation (1).
Solution: Use the modified Euler method

$$
\begin{aligned}
& t_0 = 0, \quad w_0 = y(t_0) = 0, \\
& \text{for } i = 0, N, \\
& \quad t_{i+1} = t_i + h, \\
& \quad k_1 = hf(t_i, w_i) = h(-2w_i + t_i e^{3t_i}), \\
& \quad k_2 = hf(t_{i+1}, w_i + k_1) = h\left[-2(w_i + k_1) + t_{i+1} e^{3t_{i+1}}\right] \\
& \quad w_{i+1} = w_i + \frac{1}{2}\big(k_1 + k_2\big) \\
& t_1 = t_0 + h = 0.5, \quad y(t_1) \approx w_1 = 0.5602111 \\
& t_2 = t_1 + h = 1, \quad y(t_2) \approx w_2 = 5.3014898
\end{aligned}
$$

(e) Use the Runge-Kutta method of order four with $h = 0.5$ to approximate the solution to equation (1).

Solution: Use the Runge-Kutta method of order four

$$
\begin{aligned}
& t_0 = 0, \quad w_0 = y(t_0) = 0, \\
& \text{for } i = 0, N, \\
& \quad t_{i+1/2} = t_i + h/2, \quad t_{i+1} = t_i + h, \\
& \quad k_1 = hf(t_i, w_i) = h(-2w_i + t_i e^{3t_i}), \\
& \quad k_2 = hf(t_{i+1/2}, w_i + k_1/2) = h\Big[-2(w_i + k_1/2) + t_{i+1/2} e^{3t_{i+1/2}}\Big] \\
& \quad k_3 = hf(t_{i+1/2}, w_i + k_2/2) = h\Big[-2(w_i + k_2/2) + t_{i+1/2} e^{3t_{i+1/2}}\Big] \\
& \quad k_4 = hf(t_{i+1}, w_i + k_3) = h\Big[-2(w_i + k_3) + t_{i+1} e^{3t_{i+1}}\Big] \\
& \quad w_{i+1} = w_i + \frac{1}{6}\big(k_1 + 2k_2 + 2k_3 + k_4\big) \\
& t_1 = t_0 + h = 0.5, \quad y(t_1) \approx w_1 = 0.29699975 \\
& t_2 = t_1 + h = 1, \quad y(t_2) \approx w_2 = 3.314312
\end{aligned}
$$

40 points

2. Consider the following Runge-Kutta method

$$
\begin{aligned}
& w_0 = y_0, \\
& \text{for } i = 0, 1, \cdots, N-1, \\
& \quad k_1 = hf(t_i, w_i), \\
& \quad k_2 = hf(t_i + \alpha h, w_i + \beta k_1) \\
& \quad w_{i+1} = w_i + a_1 k_1 + a_2 k_2
\end{aligned}
$$

(a) Show that the above Runge-Kutta method cannot have local truncation error $O(h^3)$ for any choice of constants a_1, a_2, α and β.

Solution:

$$
\begin{aligned}
& y(t+h) - y(t) = hy'(t) + \frac{h^2}{2} y''(t) + O(h^3) \text{ Taylor expansion} \\
& y'' = \frac{dy'}{dt} = \frac{df(t,y)}{dt} = f_t + f_y y' = f_t + f_y f \\
& \quad \Rightarrow \quad y(t+h) - y(t) = hf + \frac{h^2}{2}(f_t + f_y f) + O(h^3) \\
& \phi(t,y) = a_1 f(t,y) + a_2 f(t + \alpha h, y + \beta h f) \\
& \qquad = a_1 f + a_2 \big(f + \alpha h f_t + \beta h f_y f + O(h^2)\big) \text{ Taylor expansion} \\
& \tau(h) = \frac{y(t+h) - y(t)}{h} - \phi(t,y) \\
& \quad = (1 - (a_1 + a_2))f + h(1/2 - a_2\alpha) f_t + h(1/2 - a_2\beta) f_y f + O(h^2)
\end{aligned}
$$

For any choice of a_1, a_2, α and β, there is insufficient flexibility to match the term of h^2 in the Taylor expansions, thus the method cannot have local truncation error $O(h^3)$

(b) Show that the above Runge-Kutta method is of order 2 if, for any α,

$$\beta = \alpha, \quad a_1 = 1 - \frac{1}{2\alpha}, \quad a_2 = \frac{1}{2\alpha}.$$

Solution: To match the term of h, we need to set

$$\begin{aligned} 1 - (a_1 + a_2) = 0, \quad 1/2 - a_2\alpha = 0, \quad 1/2 - a_2\beta = 0 \\ \Rightarrow \quad \alpha \neq 0 \text{ arbitrary }, \beta = \alpha, \quad a_2 = \frac{1}{2\alpha}, \quad a_1 = 1 - \frac{1}{2\alpha} \end{aligned}$$

(c) Show that by chosing $\alpha = 1$ in (b), we obtain the modified Euler method.
Solution:

$$\alpha = 1 \quad \Rightarrow \quad \beta = 1, \quad a_1 = a_2 = \frac{1}{2}$$

The RK method is

$$w_{i+1} = w_i + \frac{h}{2}\left(f(t_i, w_i) + f(t_i + h, w_i + hf(t_i, w_i))\right)$$

which is the modified Euler method

(d) Show that by chosing $\alpha = \frac{1}{2}$ in (b), we obtain the midpoint method.
Solution:

$$\alpha = \frac{1}{2} \quad \Rightarrow \quad \beta = \frac{1}{2}, \quad a_1 = 0, \quad a_2 = 1$$

The RK method is

$$w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)\right)$$

which is the midpoint method.

20 points 3. Derive the Adams-Bashforth two step method by using the Lagrange form of the interpolating polynomial.

Solution:

$$f(t,y(t)) = p_1(t) + \frac{f''(\xi,y(\xi))}{2}(t-t_i)(t-t_{i-1}), \qquad p_1(t) \text{ linear Lagrange poly.}$$

$$p_1(t) = L_i(t)f_i + L_{i-1}(t)f_{i-1},$$

$$\int_{t_i}^{t_{i+1}} p_1(t)dt = \left(\int_{t_i}^{t_{i+1}} L_i(t)dt\right) f_i + \left(\int_{t_i}^{t_{i+1}} L_{i-1}(t)dt\right) f_{i-1}$$

$$= \frac{h}{2}\left(3f(t_i,y(t_i)) - f(t_{i-1},y(t_{i-1}))\right)$$

$$\int_{t_i}^{t_{i+1}} \frac{f''(\xi,y(\xi))}{2}(t-t_i)(t-t_{i-1})dt = \frac{5h^2}{12}f''(\mu,y(\mu)).$$

$$y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(t,y(t))dt$$

$$\Rightarrow \quad w_{i+1} = w_i + \frac{h}{2}\left(3f(t_i,w_i) - f(t_{i-1},w_{i-1})\right)$$

$$\Rightarrow \quad \tau_i(h) = \frac{5h^2}{12}f''(\mu,y(\mu)) \quad \text{for some } \mu \in [t_{i-1},t_{i+1}]$$