NUMERICAL ANALYSIS

Test 4 **Math 4365 (Spring 2012)** **April 17, 2012**

Name and ID: Solution keys

50 points 1. Apply the Nonlinear Shooting method to solve the Van del Pol equation

$$(BVP)\quad \begin{cases} y'' - \frac{1}{2}(y^2-1)y' + y = 0, & x \in (0,2), \\ y(0) = 0, \quad y(2) = 1 \end{cases}$$

and write down the detailed steps/algorithm using the 4th order Runge-Kutta method (i.e., nonlinear shooting problem, Newtow iteration, RK4 updates for IVPs, etc)

$\boxed{\text{(i) NL shooting.}}$ Find $t^* \in \mathbb{R}$ s.t t^* is the root of

$$g(t) := u(b;t) - \beta = 0$$

where $u(\cdot\,;t)$ is the solution of IVP

$$(IVP)_{NL}\ \begin{cases} u'' - \frac{1}{2}(u^2-1)u' + u = 0 \\ u(0) = 0, \quad u'(0) = t \end{cases}$$

$\Rightarrow$ The solution of (BVP) is then

$$y(x) = u(x;t^*), \quad \forall x \in [0,2].$$

$\boxed{\text{(ii) Newton Iteration.}}$

$\boxed{ALG\_NT}$

$t_0\ (=0)$ given, $\delta t_0 = \infty$

For $k = 0, 1, 2, \cdots$, until convergence (e.g. $|\delta t_k| < \varepsilon = 10^{-8}$)

$$\delta t_k = -[g'(t_k)]^{-1} g(t_k)$$

$$t_{k+1} = t_k + \delta t_k$$

where

$$g(t_k) = u(b;t_k) - \beta$$

$$g'(t_k) = w(b;t_k)$$

$t_k \to t^*$

with $w(\cdot\,;t)$, $\forall t \in \mathbb{R}$, is the solution of IVP

$$(IVP)_L\ \begin{cases} w'' - \frac{1}{2}(u^2-1)w' - \frac{1}{2}(2uw-1)u' + w = 0 \\ w(0) = 0, \quad w'(0) = 1 \end{cases}$$

(iii) RK4 updates for IVP

Let $\begin{cases} z_1 = u \\ z_2 = u' \end{cases}$ and $\begin{cases} z_3 = w \\ z_4 = w' \end{cases}$

$(IVP)_{NL} + (IVP)_L$ can be combined and rewritten to

$$\begin{cases} z_1' = z_2 \\ z_2' = \frac{1}{2}(z_1^2 - 1) z_2 - z_1 \\ z_3' = z_4 \\ z_4' = \frac{1}{2}(z_1^2 - 1) z_4 + \frac{1}{2}(2 z_1 z_3 - 1) z_2 - z_3 \\ z_1(0) = 0, \quad z_2(0) = t \\ z_3(0) = 0, \quad z_4(0) = 1 \end{cases}$$

or

$$(IVP)_c^t \begin{cases} \underline{z}' = \underline{f}(\underline{z}), \\ \underline{z}(0) = (0, t, 0, 1)^T, \end{cases} \qquad \text{with } \underline{z} \mapsto \underline{f}(\underline{z}) = \begin{bmatrix} z_2 \\ \frac{1}{2}(z_1^2 - 1) z_2 - z_1 \\ z_4 \\ \frac{1}{2}(z_1^2 - 1) z_4 + \frac{1}{2}(2 z_1 z_3 - 1) z_2 - z_3 \end{bmatrix}$$

Therefore the RK4 method applied to $(IVP)_c^t$ is

ALG-RK4

SET $z_0 = (0, t, 0, 1)^T$, $\quad h = \frac{2}{n} \quad (n \geq 1 \text{ integer})$

For $i = 0$ to $n-1$

$$\begin{aligned} \underline{k}_1 &= f(z_i) \\ \underline{k}_2 &= \underline{f}(\underline{z}_i + \tfrac{h}{2} \underline{k}_1) \\ \underline{k}_3 &= \underline{f}(z_i + \tfrac{h}{2} k_2) \\ \underline{k}_4 &= \underline{f}(\underline{z}_i + h \underline{k}_3) \\ \underline{z}_{i+1} &= \underline{z}_i + \tfrac{h}{6}(\underline{k}_1 + 2(\underline{k}_2 + \underline{k}_3) + \underline{k}_4) \end{aligned}$$

SET $g(t) := z_1^{(n)} - \beta, \quad g'(t) = z_3^{(n)}$

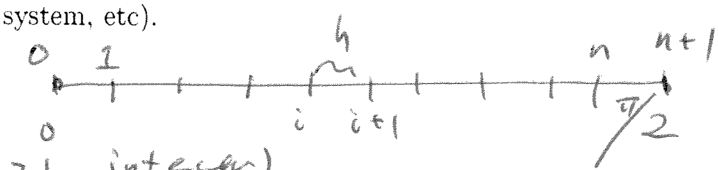
$\Rightarrow \delta t_k := \frac{1}{z_3^n}(\beta - z_1^n)$

$\boxed{\text{50 points}}$ 2. Apply the Linear Finite-Difference method to solve

$$(BVP)\begin{cases} y'' = y' + 2y + \cos x, & x \in (0, \pi/2), \\ y(0) = 0, & y(\pi/2) = 1 \end{cases}$$

and write down the detailed steps/algorithm (i.e., FD discretization, formulation of linear system, solution of linear system, etc).

(i) FD Discretization.



Set $h = \frac{(\pi/2)}{n+1}$ ($n > 0$ integer)

$x_i = ih$, $\forall i = 0, \cdots, n+1$

(BVP) is discretized by

$\forall i = 1, \cdots, n$

$$\frac{-y_{i-1} + 2y_i - y_{i+1}}{h^2} + \frac{y_{i+1} - y_{i-1}}{2h} + 2y_i = -\cos x_i$$

$$y_0 = 0, \quad y_{n+1} = 1$$

or $\forall i = 2, \cdots, n-1$

$$\left(-1 - \frac{h}{2}\right) y_{i-1} + (2 + 2h^2) y_i + \left(-1 + \frac{h}{2}\right) y_{i+1} = -h^2 \cos x_i$$

$i = 1$, $\quad (2 + 2h^2) y_1 + \left(-1 + \frac{h}{2}\right) y_2 = -h^2 \cos x_1$

$i = n$, $\quad \left(-1 - \frac{h}{2}\right) y_{n-1} + (2 + 2h^2) y_n = -h^2 \cos x_n - \left(-1 + \frac{h}{2}\right)$

(ii) Formulation of LS.

$\underline{\underline{A}}\,\underline{y} = \underline{b}$ where

with $\boxed{\begin{array}{l} \alpha = 2 + 2h^2, \\ \beta = -1 - \frac{h}{2} \\ \gamma = -1 + \frac{h}{2} \end{array}}$

$$\underset{\in \mathbb{R}^{n\times n}}{\underline{\underline{A}}} = \begin{bmatrix} \alpha & \gamma & & 0 \\ \beta & \ddots & \ddots & \\ & \ddots & \ddots & \gamma \\ 0 & & \beta & \alpha \end{bmatrix}, \quad \underset{\in \mathbb{R}^n}{\underline{b}} = \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} -h^2 \cos x_1 \\ \vdots \\ -h^2 \cos x_i \\ \vdots \\ -h^2 \cos x_n - \left(-1 + \frac{h}{2}\right) \end{bmatrix}$$

$$A = \begin{pmatrix} \beta & \ddots & \gamma \\ 0 & \beta & \alpha \end{pmatrix}$$

(iii) Solution of LS

Step 1 Factorization $\underline{A} = \underline{L}\,\underline{U}$

where $L = \begin{pmatrix} 1 & & 0 \\ l_1 & \ddots & \\ 0 & l_{n-1} & 1 \end{pmatrix}$, $U = \begin{pmatrix} u_1 & w_1 & 0 \\ & \ddots & w_{n-1} \\ 0 & & u_n \end{pmatrix}$

$u_1 = \alpha$

$\boxed{\text{ALG-FAC}}$

For $i = 1$ to $n-1$

$\quad l_i = \beta / u_i, \; w_i = \gamma$

$\quad u_{i+1} = \alpha - l_i w_i$

END

Step 2 Backward Substitution
Forward

$\underline{A}\,\underline{y} = \underline{b} \iff \underline{L}\,\underline{z} = \underline{b} \iff \begin{bmatrix} 1 & & 0 \\ l_1 & \ddots & \\ 0 & l_{n-1} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

$\phantom{\underline{A}\,\underline{y} = \underline{b} \iff{}} \underline{U}\,\underline{y} = \underline{z} \iff \begin{bmatrix} u_1 & w_1 & & 0 \\ & \ddots & w_{n-1} & \\ 0 & & u_n & \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$

$\boxed{\text{ALG-FWD}}$

$z_1 = b_1$

For $i = 2, \ldots, n$

$\quad z_i = b_i - l_{i-1} z_{i-1}$

END

$\boxed{\text{ALG-BWD}}$

$y_n = z_n / u_n$

For $i = n-1$ to 1

$$y_i = \frac{1}{u_i}\left[z_i - w_i y_{i+1} \right]$$