Midterm Exams: If you did not take your exam through CSD, then you can email yflores@math.uh.edu with the subject line "Please send me my graded midterm." Include your name and peoplesoft ID in the body of the email.

Recall: Linear Systems of Equations

Solution Process Via Elementary Row Operations

Review:

$$Ax = b$$

A: known $m \times n$ matrix

x: unknown

b: known $m \times 1$ matrix (vector)

Systems of linear equations have 0, 1 or infinitely many sol'ns.

Homogeneous vs Nonhomogeneous Linear Systems

2 types

Homog. systems → $A x = \vec{0}$ ← zero vector

(All of the rhs of equations are zero.)

Trivial sol'n.

Note: $x = \vec{0}$ always solves.

Non homog. systems → $Ax = b$ with $b \neq \vec{0}$.

Aside: Give a value of a so that the homogeneous system

matrix form

$$\begin{pmatrix} -2 & 1 & 3 \\ 1 & 1 & -1 \\ -1 & 2 & a \end{pmatrix} x = \vec{0}$$

$\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

has more than just the trivial solution.

Equation form:

$$\begin{cases} -2x_1 + x_2 + 3x_3 = 0 \\ x_1 + x_2 - x_3 = 0 \\ -x_1 + 2x_2 + ax_3 = 0 \end{cases}$$

Aug. matrix:

$$\begin{pmatrix} -2 & 1 & 3 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & 2 & a & 0 \end{pmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ -2 & 1 & 3 & 0 \\ -1 & 2 & a & 0 \end{pmatrix}$$ use

$2R_1 + R_2 \to R_2$

$R_1 + R_3 \to R_3$

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & a-1 & 0 \end{pmatrix}$$ use

$\therefore$ our system has nontrivial sol'ns iff $a = 2$.

$-R_2 + R_3 \to R_3$

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & a-2 & 0 \end{pmatrix}$$

if $a - 2 \neq 0$ then $\Rightarrow$ $x_3 = 0$, $x_2 = 0$, $x_1 = 0$ → trivial sol'n.

If $a - 2 = 0$ (i.e. $a = 2$) then infinitely many sol'ns.

ex.

$$x + ay = 0$$
$$-3x + 2y = 0$$

$$\begin{pmatrix} 1 & a & 0 \\ \boxed{-3} & 2 & 0 \end{pmatrix}$$

$3R_1 + R_2 \to R_2$

$$\begin{pmatrix} 1 & a & 0 \\ 0 & \boxed{3a+2} & 0 \end{pmatrix}$$

$3a+2 \neq 0 \Rightarrow$ only trivial sol'n.

$3a+2 = 0 \Rightarrow$ infinitely many sol'ns (i.e. there will be many nontrivial sol'ns)

$$\boxed{a = -\frac{2}{3}}$$

27. For what values of a does the system

$$x + ay - 2z = 0$$
$$2x - y - z = 0$$
$$-x - y + z = 0$$

have nontrivial solutions?

$$\begin{pmatrix} 1 & a & -2 & 0 \\ 2 & -1 & -1 & 0 \\ -1 & -1 & 1 & 0 \end{pmatrix}$$

use

$-2R_1 + R_2 \to R_2$
$R_1 + R_3 \to R_3$

$$\begin{pmatrix} 1 & a & -2 & 0 \\ 0 & -1-2a & 3 & 0 \\ 0 & a-1 & -1 & 0 \end{pmatrix}$$

use

Nobody said "use RREF"

$\frac{1}{3}R_2 + R_3 \to R_3$

$$\begin{pmatrix} 1 & a & -2 & 0 \\ 0 & -1-2a & 3 & 0 \\ 0 & \frac{1}{3}a - \frac{4}{3} & 0 & 0 \end{pmatrix}$$

If $\frac{1}{3}a - \frac{4}{3} \neq 0$ ⟹ $x_2 = 0$, $x_3 = 0$, $x_1 = 0$ (only the trivial)

If $\frac{1}{3}a - \frac{4}{3} = 0$, then we get infinitely many sol'ns (i.e. many nontrivial sol'ns)

$a = 4$

"Review"

Did you read about this?

Row Reduced Echelon Form (RREF)

→ "leading entries"

↖ 1st nonzero # in a row.

Motivating Example:

$$2x_1 + 3x_2 + 3x_3 - x_4 = 2$$
$$x_1 - 2x_2 + 4x_3 + x_4 = -1$$
$$-x_1 + 3x_3 + 2x_4 = 1$$

1. all leading entries are 1.
2. zeros above and below leading entries.
3. leading entries in a row are to the right of the leading entries in the rows above.

$$\begin{pmatrix} 2 & 3 & 3 & -1 & 2 \\ 1 & -2 & 4 & 1 & -1 \\ -1 & 0 & 3 & 2 & 1 \end{pmatrix}$$

See next page.

I used the Matrix Calculator linked from http://online.math.uh.edu

The augmented matrix is
2, 3, 3, -1, 2;
1, -2, 4, 1, -1;
-1, 0, 3, 2, 1;

R1 <-> R2 gives
1, -2, 4, 1, -1;
2, 3, 3, -1, 2;
-1, 0, 3, 2, 1;

(-2)R1 + (1)R2 -> R2 gives
1, -2, 4, 1, -1;
0, 7, -5, -3, 4;
-1, 0, 3, 2, 1;

(1)R1 + (1)R3 -> R3 gives
1, -2, 4, 1, -1;
0, 7, -5, -3, 4;
0, -2, 7, 3, 0;

(1/7)R2 -> R2 gives
1, -2, 4, 1, -1;
0, 1, -5/7, -3/7, 4/7;
0, -2, 7, 3, 0;

(2)R2 + (1)R3 -> R3 gives
1, -2, 4, 1, -1;
0, 1, -5/7, -3/7, 4/7;
0, 0, 39/7, 15/7, 8/7;

(2)R2 + (1)R1 -> R1 gives
1, 0, 18/7, 1/7, 1/7;
0, 1, -5/7, -3/7, 4/7;
0, 0, 39/7, 15/7, 8/7;

(7/39)R3 -> R3 gives
1, 0, 18/7, 1/7, 1/7;
0, 1, -5/7, -3/7, 4/7;
0, 0, 1, 5/13, 8/39;

(5/7)R3 + (1)R2 -> R2 gives
1, 0, 18/7, 1/7, 1/7;
0, 1, 0, -2/13, 28/39;
0, 0, 1, 5/13, 8/39;

(-18/7)R3 + (1)R1 -> R1 gives
1, 0, 0, -11/13, -5/13;
0, 1, 0, -2/13, 28/39;
0, 0, 1, 5/13, 8/39;

this is in rref form.

Q: What does this tell us about

$$2x_1 + 3x_2 + 3x_3 - x_4 = 2$$
$$x_1 - 2x_2 + 4x_3 + x_4 = -1$$
$$-x_1 + 3x_3 + 2x_4 = 1$$

?

A:
$$x_1 = \frac{11}{13}x_4 - \frac{5}{13}$$
$$x_2 = \frac{2}{13}x_4 + \frac{28}{39}$$
$$x_3 = -\frac{5}{13}x_4 + \frac{8}{39}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 11/13 \\ 2/13 \\ -\frac{5}{13} \\ 1 \end{pmatrix} x_4 + \begin{pmatrix} -5/13 \\ 28/39 \\ 8/39 \\ 0 \end{pmatrix}$$

inf. many sol'ns

x_4 is an arbitrary real #.

Row Reduced Echelon Form (RREF)

General Process: Apply elementary row operations to a matrix until

1. The first nonzero entry in each row (if there is one) is 1.
   (These are called leading entries.)
2. The leading entry in a row is to the right of a leading entry in a row above.
3. The entries above and below leading entries are all 0.

Examples: Which of the following augmented matrices are in RREF?

$$\begin{pmatrix} 1 & 3 & 3 & -1 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

trouble

NO

$$\begin{pmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

trouble

NO

$$\begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Yes

Example:

The RREF of the augmented matrix for the linear system

$$-x_1 + 3x_2 + x_3 - x_4 = -4$$
$$-3x_1 + 8x_2 + 4x_3 - 3x_4 = -14$$
$$4x_1 - 11x_2 - 6x_3 + 7x_4 = 20$$

is

1	0	0	-11	2
0	1	0	-3	0
0	0	1	-3	-2

Solve the system.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \\ 3 \\ 1 \end{pmatrix} x_4 + \begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

where x_4 is an arbitrary real #.

infinitely many sol'ns.

$$\left(A \;\vdots\; I_n\right) \xrightarrow{\text{RREF}} \left(I_n \;\vdots\; A^{-1}\right)$$

iff A is invertible.

Computing the Inverse of an Invertible Matrix Using RREF

↳ ??

An $n \times n$ matrix A is invertible iff there is an $n \times n$ matrix B so that

$$BA = I_n$$

← $n \times n$ identity matrix.

In this case, $AB = I_n$ and we denote $A^{-1} = B$.

↑ "A inverse"

matrix ↓ F, matrix ↓ G: FG ← (# rows F) × (# cols G)

col's F = # rows G

Typically $FG \neq GF$

When you multiply the identity matrix by another matrix, you get the other matrix.

$$C I_n = C$$

$$I_n D = D$$

Let's see if $\begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}$ is invertible. If it is, then find its inverse.

$$\left(\begin{array}{cccc} 2 & 3 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right)$$

$\downarrow$ RREF

See the video

"A is a real m×n matrix"

Question

Let $A \in R^{m,n}$ and $x \in R^n$. Denote the columns of A by $\vec{c}_1, \ldots, \vec{c}_n$ and suppose $x = (x_i)$. What is Ax?

x is a real column vector with n entries. i.e. an n×1 matrix.

$\mathbb{R}^n \equiv \mathbb{R}^{n,1}$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$A = \begin{pmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{pmatrix}$$

$$Ax = \begin{pmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{c}_1 + x_2\vec{c}_2 + \cdots + x_n\vec{c}_n$$

linear combination of $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n$

Key phrase: linear combination.

ex.

$$\begin{pmatrix} 2 & -1 & 3 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} -6 \\ 7 \\ 8 \end{pmatrix} = \begin{pmatrix} (-6)(2) + (7)(-1) + (8)(3) \\ (-6)(1) + (7)(1) + (8)(4) \end{pmatrix}$$

(columns labelled $\vec{c}_1$, $\vec{c}_2$, $\vec{c}_3$; vector labelled x)

$$= -6\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 7\begin{pmatrix} -1 \\ 1 \end{pmatrix} + 8\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

The product is a linear combination of the 3 columns in the matrix.

Important Note: You can only solve $Ax = b$ if b is a linear combination of the columns of A.

n vectors. Each vector is m×1.

Definition

Let $\vec{c}_1,\ldots,\vec{c}_n \in R^m$. The set $\{\vec{c}_1,\ldots,\vec{c}_n\}$ is linearly independent if and only if $x_1\vec{c}_1+\ldots+x_n\vec{c}_n=\vec{0}$, with $x_i \in R$, implies $x_i = 0$ for all $i=1,\ldots,n$. Otherwise, we say the set is linearly dependent.

i.e.

$$\underbrace{\begin{pmatrix}\vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix}}_{x}=\vec{0}$$

i.e. the columns of a matrix A are linearly independent if and only if the only solution to Ax = 0 is the trivial solution. Otherwise, the columns of A are linearly dependent.

Question

How is linear independence related to solving $Ax = 0$?

i.e. the columns of a matrix A are linearly independent if and only if the only solution to Ax = 0 is the trivial solution. Otherwise, the columns of A are linearly dependent.

Determine whether the set $\left\{\begin{pmatrix}-1\\2\\1\end{pmatrix},\begin{pmatrix}-2\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ is linearly independent.

1. Form $A = \begin{pmatrix}-1 & -2 & 1\\2 & 1 & 1\\1 & 0 & 1\end{pmatrix}$

2. Solve $A x = \vec{0}$.

$$\begin{pmatrix}-1 & -2 & 1 & 0\\2 & 1 & 1 & 0\\1 & 0 & 1 & 0\end{pmatrix}$$

$2R_1 + R_2 \to R_2$
$R_1 + R_3 \to R_3$

$$\begin{pmatrix}-1 & -2 & 1 & 0\\0 & -3 & 3 & 0\\0 & -2 & 2 & 0\end{pmatrix}$$

$\therefore$ inf. many sol'ns.

$\Rightarrow A x = \vec{0}$ has nontrivial sol'ns.

$\therefore$ the columns of A are linearly dependent.

i.e. $\left\{\begin{pmatrix}-1\\2\\1\end{pmatrix},\begin{pmatrix}-2\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ is linearly dependent.

Determine the value(s) of c so that $\left\{\begin{pmatrix}-1\\2\\1\end{pmatrix},\begin{pmatrix}-2c\\1\\0\end{pmatrix},\begin{pmatrix}c\\-1\\1\end{pmatrix}\right\}$ is linearly independent.

1. Form $A = \begin{pmatrix}-1 & -2c & c\\ 2 & 1 & -1\\ 1 & 0 & 1\end{pmatrix}$

2. "Solve" $A x = \vec{0}$ (Really, just determine the # of sol'ns)

$$\begin{pmatrix}-1 & -2c & c & 0\\ 2 & 1 & -1 & 0\\ 1 & 0 & 1 & 0\end{pmatrix}$$

$2R_1+R_2 \to R_2$

$R_1+R_3 \to R_3$

$$\begin{pmatrix}-1 & -2c & c & 0\\ 0 & -4c+1 & 2c-1 & 0\\ 0 & -2c & c+1 & 0\end{pmatrix}$$ use

$(2c)R_2+(-4c+1)R_3 \to R_3$

combo

$-4c+1 \neq 0$

$$\begin{pmatrix}-1 & -2c & c & 0\\ 0 & -4c+1 & 2c-1 & 0\\ 0 & 0 & -5c+1 & 0\end{pmatrix}$$

$(2c)(2c-1)+(-4c+1)(c+1) = 4c^2-2c-4c^2-4c+c+1 = -5c+1$

Case 1: If $-4c+1 \neq 0$ then we get

$$\begin{pmatrix} -1 & -2c & c & 0 \\ 0 & -4c+1 & 2c-1 & 0 \\ 0 & 0 & -5c+1 & 0 \end{pmatrix}.$$

Here we only get the trivial sol'n iff $c \neq \frac{1}{5}$.

Case 2: If $-4c+1 = 0$ (i.e. $c = \frac{1}{4}$) then we get

$$\begin{pmatrix} -1 & -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 5/4 & 0 \end{pmatrix}$$

→ Only the trivial sol'n.

Combining: we only get the trivial sol'n when $c \neq \frac{1}{5}$.

i.e. $\left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2c \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ -1 \\ 1 \end{pmatrix} \right\}$ is Linearly independent iff $c \neq \frac{1}{5}$.

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1. Determine whether the set $\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \right\}$

is linearly independent or linearly dependent.

a. linearly independent

b. linearly dependent

c. there is not enough information

$$A = \begin{pmatrix} -1 & -2 & 1 \\ -2 & -1 & -3 \\ 1 & 0 & 1 \end{pmatrix}$$

Solve $A\vec{x} = \vec{0}$

$$\begin{pmatrix} -1 & -2 & 1 & 0 \\ -2 & -1 & -3 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -2 & 1 & 0 \\ 0 & 3 & -5 & 0 \\ 0 & -2 & 2 & 0 \end{pmatrix}$$

No problem. R_2 and R_3 are not multiples of each other.

Different Problem

$$\begin{pmatrix} -1 & -2 & 1 & 0 \\ -2 & -1 & -3 & 0 \\ 1 & 5 & -6 & 0 \end{pmatrix}$$

inf. sol'ns

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Determinants

How do we take the determinant of a 1x1 matrix?

$\det(a) = a$ Just the entry

How do we take the determinant of a 2x2 matrix?

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

How do we take the determinant of an *n*x*n* matrix?

$$A = (a_{i,j}) = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}$$ much more complicated

All of the following give the same value.

1. Pick a row. (expansion across the row you pick.) Suppose it is the k^{th} row.

$$\det(A) = \sum_{j=1}^{n} (-1)^{k+j} a_{k,j} \det(A_{k,j})$$

$A_{k,j}$: Formed by removing the k^{th} row and j^{th} column from A.

2. Pick a column. (expansion down the column you pick.) Spse it is the k^{th} column.

$$\det(A) = \sum_{j=1}^{n} (-1)^{k+j} a_{j,k} \det(A_{j,k})$$

entries in the k^{th} column.

Formed by removing the j^{th} row and k^{th} column from A.

Compute the determinant of $A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \\ -3 & 0 & 4 \end{pmatrix}$ in at least 2 different ways.

← column 2 has 2 zero entries.

1. Expand down the 2^{nd} column.

$$\det(A) = 0 + (-1)^{2+2} \cdot 3 \cdot \det\begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix} + 0$$

$$= 3 \cdot (8 - -3) = \underline{\underline{33}}$$

2. Expand across row 1.

$$A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \\ -3 & 0 & 4 \end{pmatrix}$$

$$\det(A) = (-1)^{1+1} \cdot 2 \cdot \det\begin{pmatrix} 3 & 2 \\ 0 & 4 \end{pmatrix} + 0 + (-1)^{1+3} \cdot 1 \cdot \det\begin{pmatrix} -1 & 3 \\ -3 & 0 \end{pmatrix}$$

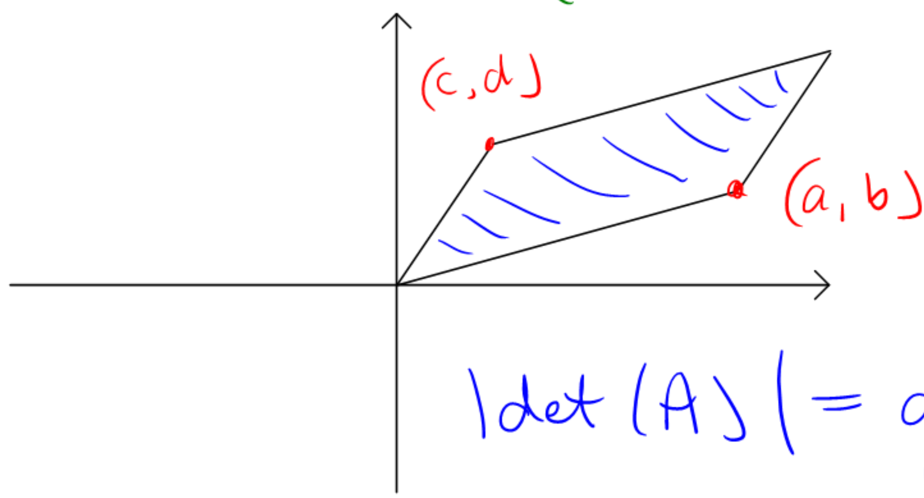
$$= 2 \cdot (12) + (0 - -9)$$

$$= 24 + 9 = \underline{\underline{33}}$$

Questions (cont.)

Is there a geometric interpretation of the determinant of an nxn matrix?

2x2 Case $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$



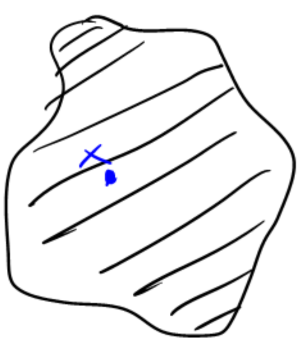
$|\det(A)|$ = area of the parallelogram.

3x3 Case: We generate a parallelopiped.

$|\det(A)|$ = volume of the parallelopiped.

In general: A nxn matrix

A acts on this set

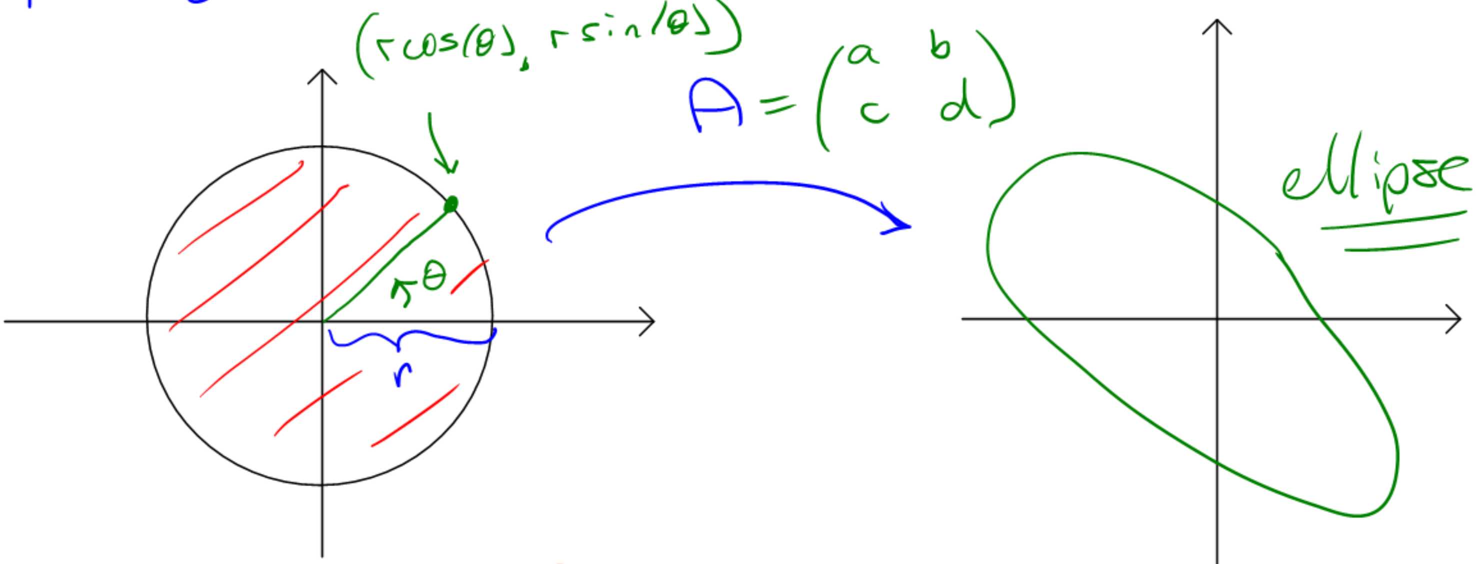


set in $\mathbb{R}^n$

volume of new set = $|\det(A)|$ times volume of original set.

new set in $\mathbb{R}^n$

Spse your set is a circle.



Area $= \pi r^2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \end{pmatrix} = \begin{pmatrix} ar\cos(\theta) + br\sin(\theta) \\ cr\cos(\theta) + dr\sin(\theta) \end{pmatrix}$$

You should

1. watch the portion of the video posted earlier this week dealing with determinants.

There is a link between determinants and ERO.

2. watch the portion of video dealing with eigenvalues and eigenvectors + the supplemental video.

Question

How do elementary row operations effect the determinant of a matrix?

Question

How is the determinant of a matrix related to the determinant of the transpose of the matrix? Of the inverse of the matrix (if it exists)?

Question

Suppose A and B are nxn matrices and α is a scalar. Determine which of the following are true:

- $\det(A + B) = \det(A) + \det(B)$
- $\det(A\ B) = \det(A)\ \det(B)$
- $\det(\alpha\ A) = \alpha\ \det(A)$

Questions (cont.)

How is the idea of determinant related to linear independence?

Definition

An nxn matrix *A* is **nonsingular** if and only if

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2. Give the determinant of $\begin{pmatrix} 2 & 1 & 1 \\ -1 & -1 & 2 \\ -3 & 0 & -2 \end{pmatrix}$.

Eigenvalues and Eigenvectors

Definition – Part 1

Suppose $A \in R^{n \times n}$. We say that a number $\lambda \in R$ is a real eigenvalue of A if and only if there is a nonzero vector $x \in R^n$ so that $Ax = \lambda x$. In this case, the vector x is referred to as an eigenvector associated with the real eigenvalue λ.

Definition – Part 2

Suppose $A \in R^{n \times n}$. We say that a number $\lambda \in C$ with $im(\lambda) \neq 0$ is a complex eigenvalue of A if and only if there is a nonzero vector $x \in C^n$ so that $Ax = \lambda x$. In this case, the vector x is referred to as an eigenvector associated with the complex eigenvalue λ.

Question: How do we find the eigenvalues and associated eigenvectors of a real square matrix A?

Definition: Characteristic Polynomial

Example

Find the eigenvalues and associated eigenvectors for the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

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3. Give the smallest eigenvalue of the matrix $\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$.

4. Give the largest eigenvalue of the matrix $\begin{pmatrix} 3 & 2 \\ 0 & 2 \end{pmatrix}$.

Example

Find the eigenvalues and associated eigenvectors for the matrix $A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$.

Example

Find the eigenvalues and associated eigenvectors

for the matrix $A = \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & 1 \\ -3 & 2 & -2 \end{pmatrix}$.