Chapter 3: The Real Numbers

1. Overview

In one sense real analysis is just doing calculus all over again, only this time we prove everything. But in another larger sense this class is much more than that. It's about setting up a system to analyze things like calculus thoroughly and rigorously so that we can move beyond calculus.

Our system is built up on the axiomatic assumptions (or definitions) on the real numbers.

So, what is a real number? In mathematics, the real numbers may be described informally as numbers that can be given by an infinite decimal representation, such as 2.4871773339. The real numbers include both rational numbers, such as 42 and -23/129, and irrational numbers, such as π and $\sqrt{2}$, and can be represented as points on an infinitely long number line.

A more rigorous definition of the real numbers was one of the most important developments of 19th century mathematics. In this book, we use an **axiomatic definition** (i.e. we assume these properties **automatically** hold) of the real numbers as the unique **complete Archimedean ordered field**, i.e. we **assume** that **R** has the following mathematical structures (click here to see more) :

(a) (**Archimedean fields**): There are two operations "+" and "·" on **R**, i.e. we can do addition and multiplication for any two real numbers;

(b)(**ordered**): There is also an *order* or *relation* " $<$ " on **R**, i.e. we can compare two real numbers to see which one is bigger;

(c) (**complete**): Every nonempty subset S of **R** that is bounded above has a **least** upper bound. That is, $\sup S$ exists and is a real number (i.e. the set **R** is big enough, it contains $\sup S$ for every subset S).

Note that **axioms** are some starting assumptions from which other statements are logically derived. Unlike theorems, axioms can not be derived by principles of deduction nor demonstrable by formal proofs, simply because they are starting assumptions and there is nothing else they logically follow from (otherwise they would be called theorems). Based on these axioms, more theorems are derived in Section 12.

Using " $<$ ", we can define $|x|$, the *absolute values of* x). Using the absolute value, we have the concept of the *distance* of two real numbers. The distance concept allows us to define the *neighborhood* (see section 13, P. 129). Then we can introduce the concepts of *interior point*, *boundary point*, *open set*, *closed set*, ..etc.. (see Section 13: *Topology of the reals*). All these concepts have something to do with the **distance**, which describes how close two points are. These concepts will be used in the study of limit, continuity, ...etc..

Finally, we introduce the concept of *compact set* (see section 14). A set S in $\mathbf{R}$ is compact if and only if it is closed and bounded. The most important property for a **compact** set S is that *every open cover (usually it contains* **infinitely** *many open sets) has a finite subcover.* This **important property** allows us to pass from **infinitely many** to **finitely many** (this trick will be used quite often in the proofs of this course).

Section 10: Induction

When a statement involves **natural numbers** (i.e. non-negative integers), you usually use the method of *induction* (click here to see more) to prove it. The procedure goes as follows: **Step 1**: Verify the statement holds for $n = 1$ (base step); **Step 2**: Assume that, for each natural number n, the statement holds for n (Induction hypothesis, or IH), try to verify the statement also holds for $n + 1$.

Example. *Prove that*

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$

Proof: We use induction on n.
Base Step: $n = 1$. Then

$$\frac{1}{2!} = \frac{1}{2} = 1 - \frac{1}{1+1}.$$

Inductive Step: Assume that $n \geq 1$ and that (here (IH) means the "induction hypothesis")

$$\text{(IH)} \quad \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$

Then

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n+1}{(n+2)!} \quad = \quad \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} + \frac{n+1}{(n+2)!}$$

$$
\begin{aligned}
&= 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!} \quad \text{by (IH)} \\
&= = 1 - \frac{1}{(n+1)!}\left(1 - \frac{n+1}{n+2}\right) \\
&= 1 - \frac{1}{(n+1)!}\left(\frac{1}{n+2}\right) \\
&= 1 - \frac{1}{(n+2)!}.
\end{aligned}
$$

Hence, by induction, we have proved that

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$

Section 11, Ordered Fields

This section explains that is a **field** and what an **order** is.

When we say that $\mathbf{R}$ is a **field**(click here to see more), we mean that there are two operations "+" (addition) and "·"(multiplication) on $\mathbf{R}$ which satisfy 11 properties (see Page 108-109, A1-DL).

By the **ordered field**(click here to see more), we mean that, in addition to two operations "+" (addition) and "·"(multiplication) on $\mathbf{R}$, there is also an *order* or *relation* " $<$ " which satisfy Q1-Q4 properties on Page 109. We assume these properties at the beginning which are called *axioms*, i.e. we admit them automatically without proofs.

Based on these *axioms*, this chapter continues derives **Theorems**. see, for example, Theorem 11.1, Theorem 11.7. Read **Practice 11.2-11.6** and **Example 11.5** carefully to see how to use these axioms and theorems to do the proofs(helpful in doing HWs).

On Page 113, it introduces the concept of *absolute value* by using the order "¡". Then it derives the properties of the absolute value. The concept of absolute value is essential in section 13.

Note that in doing HW, **what you can use freely** are those assumptions in the axiom (of ordered field) plus the theorems or results which you have already proved. See the following example.

11.3(a): *Show that* $-(-x) = x$.
Proof:

$$(-x) + (-(-x)) = 0 = x + (-x) \qquad \text{by Axiom A5}$$

$$(-x) + (-(-x)) = (-x) + x \qquad \text{by Axiom A2}$$

$$x + ((-x) + (-(-x))) = x + ((-x) + x) \qquad \text{by Axiom A1}$$

$$(x + (-x)) + (-(-x)) = (x + (-x)) + x \qquad \text{by Axiom A3}$$

$$0 + (-(-x)) = 0 + x \qquad \text{by Axiom A5}$$

$$(-(-x)) + 0 = x + 0 \qquad \text{by Axiom A2}$$

$$-(-x) = x \qquad \text{by Axiom A4.}$$

Section 12: The Completeness Axiom

Note that the axioms about "+","·" and "<" does not fully characterize **R**. We need another **important** axiom on **R** in section 12, called **completeness axiom** as follows: *Every nonempty subset* S *of* **R** *that is bounded above has a least upper bound. That is,* $\sup S$ *exists and is a real number.* This means that the set **R** is big enough, it contains all $\sup S$ for every subset S. So the **full** characterization for **R** is that **R** is a **complete ordered field** (click here to see more),

First of all, in this section, you need to make clear what are the meanings of **upper bound**, **least upper bound**(or **supremum** (click here to see more), **greatest lower bound**(or **infimum** (click here to see more), **maximum** (the **achieved** upper bound, i.e. it is the (least) upper bound which is also **in** the set S), and **minimum** (also **achieved**).

Here are some highlights:

- 1. The axiom of **R** is **complete** is essential, i.e. for **every** subset S of **R** which is **bounded above**, $\sup S$ **always exists**!!!. Note, **Q**, the set of all **rational numbers**, however, does not have such property, for example, let $S = \{q \in \mathbf{Q} \mid 0 \le q \le \sqrt{2}\}$, then $\sup S$ does not exist in **Q**. In fact, $\sup S = \sqrt{2}$ is not a rational number. This shows that **Q** is **not** bigger enough!, you now see why we need to study **R**, rather than **Q**.

- 2. Although $\sup S$ always exists, for all subset S of **R** which is bounded above, $\sup S$ may not be in the set S. For example, let $S = \{x \in \mathbf{R} \mid 0 \le x < \sqrt{2}\}$, then $\sup S = \sqrt{2}$, however, $\sqrt{2} \notin S$. If $\sup S$ is in S, then we call it **maximum** of S, denoted by $\max S$.

- 3. If S is a **finite** set, then $\max S$ and $\min S$ always exist. This property is **frequently** use in section 14 (the concept of **compactness** allows us pass from **infiniteness** to **finiteness**.

  The last part of the section deals with the **density** of rational numbers in the real numbers, i.e. for every two real numbers $x < y$, there is a **rational number** r such that $x < r < y$.

Section 13: The Topology of the Reals

By the **topology** of **R**, we mean the **collection** (or the **set**) of all open subset of **R**. Hence, you see that the main purpose of this section is to introduce the concept of **open set** (of course, as well as other notions: **interior point**, **boundary point**, **closed set**, **open set**, **accumulation point** of a set S, **isolated point** of S, the **closure** of S, etc.).

The approach is to use the **distance** (or absolute value). First, it introduce the concept of **neighborhood** of a point $x \in \mathbf{R}$ (denoted by $N(x, \epsilon)$ see (page 129)(see also the **deleted neighborhood**). It is the most **convenient** concept (think about what does **your neighbor** mean: it means someone lives within your **distance**). It then introduces **interior point** (i.e. if you are surrounded by your neighbors, then **you** are the **interior point**), **boundary point**, ...

Read the examples and try to do exercises to see (1): how to determine whether it is open, close, interior points, etc., (2) How to write rigorous proofs once you conclude your answer.

One of the most important properties is: *(a) The union of any collection (can be* **infinite***) of open sets is an open set, (b) The intersection of any* **finite** *collection of open sets is an open set.* The **topology** of $\mathbf{R}$ refers to the collection of all open sets of $\mathbf{R}$ (i.e., in the abstract setting: A **topological space** X is a set with a collection of subsets, denoted $\mathcal{T}$(elements in $\mathcal{T}$ are called open sets), such that (a) and (b) holds).

For the **closed** set, we have the following properties: (a) The **finite** union of any collection of closed sets is a closed set, (b) The intersection of any collection (can be **infinite**) of closed sets is closed set.

Try to use the **terms** we introduced to do some proofs.

11.9(a): *Prove that an accumulation point of a set S is either an interior point of S or a boundary point of S.*
Proof: Let $x \in \mathbf{R}$ be an accumulation point of S, and assume that x is not an interior point of S(otherwise we are done). We need to show that x is boundary point of S. So let $\epsilon > 0$; we need to show that $N(x;\epsilon) \cap S \neq \emptyset$ and $N(x;\epsilon) \cap (\mathbf{R} \backslash S \neq \emptyset$. On the one hand, since x is an accumulation point of S, we know by definition that $N^*(x;\epsilon) \cap S \neq \emptyset$; since $N^*(x;\epsilon) \subset N(x;\epsilon)$, it follows that $N(x;\epsilon) \cap S \neq \emptyset$. On the other hand, since x is not an interior point of S, we cannot have $N(x;\epsilon) \subset S$; that is $N(x;\epsilon) \cap (\mathbf{R} \backslash S \neq \emptyset$. This proves our claim.

Section 14: Compact Sets

A set S is **compact** (click here to see more),if and only if *every open cover* $\mathcal{F}$ *(usually it contains* **infinitely** *many open sets) has a finite subcover* $\mathcal{G}$. This **important property** allows us to pass from **infinitely many** to **finitely many**. Almost all proofs of this section uses this **important** property.

The set $(0,1]$ is not compact (it is not closed), for example, the $\{(1/n, 1+(1/n))\}$ is an open cover for $(0,1]$ but you can **not** find a **finite** sub-cover.

As I mentioned, the important property of compactness allows us to pass from **infinitely many** to **finitely many**. In practice, how to do it? **Step 1**, **construct** an open cover, **step 2** Use the compactness to pass from **infinitely** cover to a **finitely** subcover, and see how to derive your conclusion by using the **finiteness** property. Note the proof is often combined with the **method of proof by contradiction** (see the proof of Theorem 14.7 on P. 142).

As a warm up, the proof of the statement that *If a set* S *in* **R** *is compact, then it is bounded* goes as follows: We can cover the set S by $I_n = (-n, n)$, i.e. $S \subset \cup_{n=1}^{\infty} I_n$. Since S is compact, this open cover $\{I_n\}$ has a finite subcover, i.e. there exist finite many integers $n_1, \ldots, n_k$ such that

$$S \subset (I_{n_1} \cup \cdots \cup I_{n_k}.$$

Let $m = \max\{n_1, \ldots, n_k\}$ (note for a **finite** set, a maximum always exists!!!). Hence $S \subset (-m, m)$. SO S is bounded. Note, you should learn from this proof how to: **Step 1**, **construct** an open cover, **step 2** Use the compactness to pass from **infinitely** cover to a **finitely** subcover, and see how to derive your conclusion by using the **finiteness** property.

Another warm up: try to prove the statement that *If a set* S *in* **R** *is compact, then it is closed.*

Of course, Heine-Borel (see Theorem 14.5) (click here to see more), gives an important characterization: *A set* S *in* **R** *is compact if and only if it is closed and bounded.*