**Math 6396, Riemannian Geometry(Theory of Surfaces)**

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# 1 Surfaces

- **Parameterized Surfaces**: A *parameterized surface* is a map $\mathbf{x}$ from an (connected) open subset $U$ of $\mathbf{R}^2$ to $\mathbf{R}^3$. We can write $\mathbf{x}(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v))$, where $(u, v) \in U$. Here variables $u, v$ are called the *parameters*, they are coming from a (connected) open subset $U$ of $\mathbf{R}^2$. If $\mathbf{x}$ is differentiable, the surface is called a **differentiable surface**.

- **Regular Parameterized Surface**: A parameterized surface $\mathbf{x} : U \to \mathbf{R}^3$ is said to be *regular* if it is smooth(its partial derivatives are continuous) and the vectors $\mathbf{x}_u, \boldsymbol{\sigma}_v$ are linearly independent at all points $(u, v) \in U$(or equivalently, $\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$ for all points $(u, v) \in U$), where $\mathbf{x}_u, \mathbf{x}_v$ are the partial derivatives of $\mathbf{x}$.

- **Tangent Space**: Let $S$ be a regular surface. The tangent space $T_P(S)$ of $S$ at $P$ is the set of vectors which arise as velocity vectors(the derivative of the curve) of curves on $S$. The following is the precise definition.

  **Definition.** *Let $\mathbf{x} : U \to \mathbf{R}^3$ be a parametrized surface(we call this surface $S$). Let $P \in S$. We say a vector $\mathbf{v}_P$ is* **tangent to** *$S$ at the point $P$ if $\mathbf{v}_P$ is the tangent vector of some curve on $S$. That is, there is some $\boldsymbol{\alpha} : I \to S$ with $\boldsymbol{\alpha}(0) = P, \boldsymbol{\alpha}'(0) = \mathbf{v}_P$. Usually we write $\mathbf{v}$ instead of $\mathbf{v}_P$ when no confusion will arise. The* **tangent space (plane)** *of $S$ at $P$ is defined to be*

$$T_P(S) = \{\mathbf{v} \mid \mathbf{v} \textit{ is tangent to } S \textit{ at } P\}.$$

Let $\mathbf{x} : U \to \mathbf{R}^3$ be a parametrized surface(we call this surface $S$). There are two (special) curves on $S$ which passing through $P$ are so-called *u-parameter* curve (resp. *u-parameter* curve) $\boldsymbol{\alpha}(u) = \mathbf{x}(u, v_0)$ (resp. $\boldsymbol{\alpha}(v) = \mathbf{x}(u_0, v)$). Their velocity vectors are $\mathbf{x}_u, \mathbf{x}_v$. They are linearly independent. Hence, $\{\mathbf{x}_u, \mathbf{x}_v\}$ form a *basis* of the vector space $T_P(S)$, i.e. $\mathbf{v} \in T_P(S)$ *if and only if* $\mathbf{v} = a\mathbf{x}_u + b\mathbf{x}_v$, *where* $\mathbf{x}_u, \mathbf{x}_v$ *are evaluated at* $(u_0, v_0)$. This best describes what the tangent space $T_P(M)$ looks like.

- **Normal Vectors**. The unit vector

$$\mathbf{n}_P = \frac{\mathbf{x}_u(u_0, v_0) \times \mathbf{x}_v(u_0, v_0)}{\|\mathbf{x}_u(u_0, v_0) \times \mathbf{x}_v(u_0, v_0)\|}$$

is *(unit) normal vector to the surface $S$ at the point $P$.*

# 2 The First Fundamental Form, Length and Surface Area

- **Inner Product**: Recall that an inner(dot) product on $\mathbf{R}^3$ determines the length(norm) of vectors in $\mathbf{R}^3$ and the angle of vectors in $\mathbf{R}^3$ as follows: $|\mathbf{v}|^2 = <\mathbf{v}, \mathbf{v}>$ and $\cos\theta = \dfrac{<\mathbf{v}, \mathbf{v}>}{|\mathbf{v}||\mathbf{w}|}$.

- **Length**: Let $\mathbf{x} : U \to \mathbf{R}^3$ be a parametrization of a surface $S$. Its *first fundamental form* or *metric* is

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

where $E =< \mathbf{x}_u, \mathbf{x}_u >, F =< \mathbf{x}_u, \mathbf{x}_v >, G =< \mathbf{x}_v, \mathbf{x}_v >$. It is an **intrinsic** quantity that it relates to measurements (of length, area, and angle etc.) inside the surface.

The length of a curve $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$ in the surface $S$ from $\boldsymbol{\alpha}(t_0)$ to $\boldsymbol{\alpha}(t_1)$ is given by

$$L = \int_{t_0}^{t_1} \left| \frac{d\mathbf{x}}{dt} \right| dt = \int_{t_0}^{t_1} |\mathbf{x}_u u' + \mathbf{x}_v v'| \, dt$$

$$= \int_{t_0}^{t_1} \sqrt{< \mathbf{x}_u, \mathbf{x}_u > u'^2 + 2 < \mathbf{x}_u, \mathbf{x}_v > u'v' + < \mathbf{x}_v, \mathbf{x}_v > v'^2} \; dt$$

$$= \int_{t_0}^{t_1} \sqrt{Edu'2 + 2Fu'v' + Gv'^2} \; dt.$$

- **Surface Area**: Let $R \subset S$ be a bounded region (the image of $\mathbf{x} : Q \to \mathbf{R}^3$, where $Q \subset U$) of a regular surface contained in the coordinate neighborhood of the parametrization $\mathbf{x} : U \to \mathbf{R}^3$. Then the surface area of $R$ is given by

$$\text{Area of } R = \int\int_R d\sigma = \int\int_Q \sqrt{EG - F^2} \; dudv.$$

# 3 The Gauss map, the Shape Operator and the Second Fundamental Form

- **The Directional Derivatives**: Let $S \subset \mathbf{R}^3$ be a surface and let $g(x, y, z)$ be a function defined on $S \subset \mathbf{R}^3$, Let $\mathbf{v} \in T_P(S)$, then $\mathbf{v}$ is the velocity vector of some curve $\boldsymbol{\alpha}$ on $S$, i.e. $\boldsymbol{\alpha}(0) = P$, $\boldsymbol{\alpha}'(0) = \mathbf{v}$. We define, for $\mathbf{v} \in \mathbf{T_P(M)}$, the *directional derivative of $g$ in the $\mathbf{v}$-direction* by

$$\bigtriangledown_{\mathbf{v}} g(P) = \frac{d}{dt}(g(\boldsymbol{\alpha}(t)))|_{t=0} = \bigtriangledown g(\boldsymbol{\alpha}(t)) \cdot \mathbf{v}.$$

This definition only depends on $g$ and $\mathbf{v}$, independent of the choice of $\boldsymbol{\alpha}$.

Given a parametrization $\boldsymbol{\sigma} : U \to M$. Write $g(u, v) = g(\boldsymbol{\sigma}(u, v))$, then $g$ is (viewed) as a function of $u, v$. Hence, we may write

$$g_u = \frac{\partial(g(\boldsymbol{\sigma}(u, v))}{\partial u}, \quad g_v = \frac{\partial(g(\boldsymbol{\sigma}(u, v))}{\partial v}.$$

Then, since $\frac{d}{du}(\boldsymbol{\sigma}(u, v_0)) = \boldsymbol{\sigma}_u$, we have, by definition (taking $\boldsymbol{\alpha}(t) = \boldsymbol{\sigma}(t + u_0, v_0)$),

$$\bigtriangledown_{\boldsymbol{\sigma}_u} g = \frac{d}{du}(g(\boldsymbol{\sigma}(u, v_0))|_{u=u_0} = \frac{\partial g}{\partial u}|_{u=u_0}.$$

Similarly,

$$\bigtriangledown_{\boldsymbol{\sigma}_v} g = \frac{\partial g}{\partial v}.$$

For a vector valued function(called vector field over $S$) $\mathbf{g} = (g_1, g_2, g_3) : S \to \mathbf{R}^3$, we define, for $\mathbf{v} \in T_P(M)$, the *directional derivative of* $\mathbf{g}$ *in the* $\mathbf{v}$*-direction* by

$$\bigtriangledown_{\mathbf{v}} \mathbf{g} = (\bigtriangledown_{\mathbf{v}} g_1, \bigtriangledown_{\mathbf{v}} g_2, \bigtriangledown_{\mathbf{v}} g_3).$$

Also, with a parametrization $\boldsymbol{\sigma} : U \to M$, write $\mathbf{g}(u, v) = \mathbf{g}(\boldsymbol{\sigma}(u, v))$. Then,

$$\bigtriangledown_{\boldsymbol{\sigma}_u} g = \frac{\partial g}{\partial u}, \qquad \bigtriangledown_{\boldsymbol{\sigma}_v} g = \frac{\partial g}{\partial v}.$$

- **The Gauss Map**: The **Gauss map** of $S$ is the map $\mathbf{n} : S \to S^2 \subset \mathbf{R}^3$, which sends every point $P \in S$ to the unit normal $\mathbf{n}_P$ to the surface $S$ at the point $P$.

- **Shape Operator**:

**Theorem.** *For any* $\mathbf{v} \in T_P(S)$*, the directional derivative* $\bigtriangledown_{\mathbf{v}} \mathbf{n}(P) \in T_P(S)$*. Moreover, the linear map* $S_P : T_P(S) \to T_P(S)$ *defined by*

$$S_P(\mathbf{v}) = - \bigtriangledown_{\mathbf{v}} \mathbf{n}(P)$$

*is a* **symmetric** *linear map, where* $\bigtriangledown_{\mathbf{v}} \mathbf{n}(P)$ *is the directional derivative of* $\mathbf{n}$ *along the direction* $\mathbf{v}$*. Here that* $S_P$ *is symmetric means that, for any* $\mathbf{u}, \mathbf{v} \in T_P(M)$*, we have*

$$S_P(\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot S_P(\mathbf{v}).$$

**Definition.** $S_P : T_P(S) \to T_P(S)$ *defined by*

$$S_P(\mathbf{v}) = - \bigtriangledown_{\mathbf{v}} \mathbf{n}(P)$$

*is called the* **shape operator** *of* $S$ *at* $P$*.*

- **The Second Fundamental Form**:

**Definition.** *Let $S$ be a regular surface. The* **second fundamental form***, denoted by $II$, assigns, for every $P \in S$, a map $II_P : T_P(S) \times T_P(S) \to \mathbf{R}$ defined by $II_P(\mathbf{u}, \mathbf{v}) = \mathbf{v} \cdot S_P(\mathbf{u})$, for $\mathbf{u}, \mathbf{v} \in T_P(S)$.*

In terms of local parameterization $\mathbf{x} : U \to S \subset \mathbf{R}^3$ of the surface $S$, $\mathbf{u} = a\mathbf{x}_u + b\mathbf{x}_v$, $\mathbf{v} = c\mathbf{x}_u + d\mathbf{x}_v$. Hence, by linearity,

$$II_P(\mathbf{u}, \mathbf{v}) = acII_P(\mathbf{x}_u, \mathbf{x}_u) + (bc + ad)II_P(\mathbf{x}_u, \mathbf{x}_v) + bdII_P(\mathbf{x}_v, \mathbf{x}_v).$$

Write $e = II_P(\mathbf{x}_u, \mathbf{x}_u), f = II_P(\mathbf{x}_u, \mathbf{x}_v), g = II_P(\mathbf{x}_v, \mathbf{x}_v)$. Then

$$II_P(\mathbf{u}, \mathbf{v}) = e(ac) + f(bc + ad) + g(bd).$$

Hence, the second fundamental form $II$ only depends on the data $\{e, f, g\}$, we sometimes also call the data $\{e, f, g\}$ the second fundamental form, if no confusion arises. By calculation,

$$e = -\mathbf{x}_u \cdot \mathbf{n}_u = \mathbf{n} \cdot \mathbf{x}_{uu}, \quad f = -\mathbf{n}_u \cdot \mathbf{x}_v = \mathbf{n} \cdot \mathbf{x}_{uv}, \quad g = -\mathbf{x}_v \cdot \mathbf{n}_v = \mathbf{n} \cdot \mathbf{x}_{vv}.$$

- **The matrix of the shape operator** $S_P$:

The matrix of the shape operator $S_P$ with respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ is

$$A = -\mathcal{F}_I^{-1}\mathcal{F}_{II},$$

where

$$\mathcal{F}_I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad \mathcal{F}_{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix}.$$

If we write

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

i.e.

$$S_P(\mathbf{x}_u) = a\mathbf{x}_u + b\mathbf{x}_v$$

$$S_P(\mathbf{x}_v) = c\mathbf{x}_u + d\mathbf{x}_v$$

then

$$a = -\frac{fF - eG}{EG - F^2}, \quad b = -\frac{eF - fE}{EG - F^2},$$
$$c = -\frac{gF - fG}{EG - F^2}, \quad d = -\frac{fF - gE}{EG - F^2}.$$

# 4 Curvatures

- **The Normal Curvature of Curves on a Surface:**

Let $\mathbf{v} \in T_P(S)$ be a unit vector. We slice the surface $S$ with the plane through $P$ spanned by $\mathbf{n}(P)$ (the unit-normal at $P$) and a *unit* vector $\mathbf{v} \in T_P(M)$. Let $\boldsymbol{\alpha}$ be the arc-length-parametrized curve obtained by taking such slice. We have such that $\boldsymbol{\alpha}(0) = P, \boldsymbol{\alpha}'(0) = \mathbf{v}$. $\boldsymbol{\alpha}'$ (evaluated at 0) is perpendicular to $\mathbf{n}$(the normal to the surface $S$), so $\{\boldsymbol{\alpha}', \mathbf{n}, \mathbf{n} \times \boldsymbol{\alpha}'\}$ are mutually perpendicular unit vector(called an orthonormal basis). Since $\boldsymbol{\alpha}''$ is perpendicular to $\boldsymbol{\alpha}'$ (use the unit vector trick!), $\boldsymbol{\alpha}''$ is a linear transformation of $\mathbf{n}, \mathbf{n} \times \boldsymbol{\alpha}'$, i.e.

$$\boldsymbol{\alpha}'' = \kappa_n \mathbf{n} + \kappa_g \mathbf{n} \times \boldsymbol{\alpha}'.$$

The scalars $\kappa_n$ and $\kappa_g$ are called the *normal curvature* and the *geodesic curvature* of $\boldsymbol{\alpha}$.

We have

$$\kappa_n = II_P(\mathbf{v}, \mathbf{v}) = \mathbf{v} \cdot S_P(\mathbf{v})$$

where $II_p$ is the second fundamental form of $S$(so the normal curvature can be defined by the second fundamental form).

- **The Principal Curvatures of a Surface**: The eigenvalues of the shape operator $S_P$ are called the *Principal curvatures* of $S$.

  Since the matrix of the shape operator $S_P$ with respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ is

$$A = -\mathcal{F}_I^{-1}\mathcal{F}_{II},$$

  we have that *the principal curvatures $\kappa_1, \kappa_2$ of $S$ are the roots of the equation*

$$\det(\mathcal{F}_{II} - \kappa\mathcal{F}_I) = 0.$$

  *The corresponding principal directions are non-zero $2 \times 1$ column matrix $T$ such that*

$$(\mathcal{F}_{II} - \kappa\mathcal{F}_I)T = 0.$$

  Let $\mathbf{e}_1, \mathbf{e}_2$ be unit vectors in the principal directions at $P$ corresponding principal curvatures $\kappa_1, \kappa_2$. Then $\mathbf{v} = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$ for some $\theta \in [0, 2\pi)$. Hence

$$II_P(\mathbf{v}, \mathbf{v}) = S_P(\mathbf{v}) \cdot \mathbf{v} = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$$

  This shows that *the principal curvatures are the maximum and minimum (singed) curvature of the various normal slices.*

- **The Gauss Curvature and Mean Curvature:** Let $\kappa_1, \kappa_2$ be the eigenvalues of the shape operator $S_P$. Then

$$K = \kappa_1\kappa_2$$

  is called the *Gaussian curvature* of $S$ and

$$H = \kappa_1 + \kappa_2$$

  is called the *mean curvature* of $S$.

$$K = \frac{eg - f^2}{EG - F^2},$$
$$H = \frac{eG - 2fF + gE}{2(EG - F^2)}.$$

# 5 Gauss Curvature, Gauss equation and the Codazzi-Mainardi equations

- **Gauss Curvature**: Recall that the Gauss curvature is given by

$$K = \frac{eg - f^2}{EG - F^2}.$$

Although the above formula involves the second fundamental form, Gauss theorem egregium tells us that we can actually calculate the Gauss curvature in terms of $E$, $F$ and $G$ only, i.e. we can get a formula which only involves the first fundamental form. So the Gauss curvature is, in fact, an *intrinsic quantity*(which depends on the surface only).

- **Christoffel symbols**: By expressing the derivatives of the vectors $\mathbf{x}_u, \mathbf{x}_v$ and $\mathbf{n}$ in the basis $\{\mathbf{x}_u, \mathbf{x}_v, \mathbf{n}\}$, we obtain

$$\mathbf{x}_{uu} = \Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v + e\mathbf{n},$$
$$\mathbf{x}_{uv} = \Gamma^1_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v + f\mathbf{n},$$
$$\mathbf{x}_{vu} = \Gamma^1_{21}\mathbf{x}_u + \Gamma^2_{21}\mathbf{x}_v + f\mathbf{n},$$
$$\mathbf{x}_{vv} = \Gamma^1_{22}\mathbf{x}_u + \Gamma^2_{22}\mathbf{x}_v + g\mathbf{n},$$
$$\mathbf{n}_u = a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v,$$
$$\mathbf{n}_v = a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v,$$

where

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \mathcal{F}_{II}\mathcal{F}_I^{-1}.$$

The six functions $\Gamma^l_{ik} = \Gamma^l_{ki}, 1 \leq i, l, k \leq 2$, are called the **Christoffel symbols**.

To compute the Christoffel symbols, we take the inner product of the first four relations with $\mathbf{x}_u$ and $\mathbf{x}_v$, for example, in the first two relations(note that $\mathbf{x}_u \cdot \mathbf{n} = \mathbf{x}_v \cdot \mathbf{n} = 0$), we get

$$\Gamma^1_{11}E + \Gamma^2_{11}F =< \mathbf{x}_{uu}, \mathbf{x}_u >= \frac{1}{2}E_u,$$

$$\Gamma^1_{11}F + \Gamma^2_{11}G =< \mathbf{x}_{uu}, \mathbf{x}_v >= F_u - \frac{1}{2}E_v.$$

Solving the above system of linear equations, we get $\Gamma^l_{11}$ and $\Gamma^2_{11}$. Other Christoffel symbols can be computed in a similar way:

$$\Gamma^1_{12}E + \Gamma^2_{12}F = \frac{1}{2}E_v,$$

$$\Gamma^1_{12}F + \Gamma^2_{12}G = \frac{1}{2}G_u.$$

$$\Gamma^1_{22}E + \Gamma^2_{22}F = F_v - \frac{1}{2}G_u,$$

$$\Gamma^1_{22}F + \Gamma^2_{22}G = \frac{1}{2}G_v.$$

Note, the term $\Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v$ represents the orthogonal projection of $\mathbf{x}_{uu}(p)$ to the tangent space $T_p(S)$, which will be called the *covariant derivative* of $\mathbf{x}_u$ in the direction $\mathbf{x}_u$, we will denote it by

$$D_{\mathbf{x}_u}\mathbf{x}_u = \Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v.$$

Other covariant derivatives are defined in a similar way. We have

$$\Gamma^1_{11} = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}, \qquad \Gamma^2_{11} = \frac{2EF_u - FE_v - FE_u}{2(EG - F^2)}$$

$$\Gamma^1_{12} = \frac{GE_v - FG_u}{2(EG - F^2)}, \quad \Gamma^2_{12} = \frac{EG_u - FE_v}{2(EG - F^2)}$$
$$\Gamma^1_{22} = \frac{2GF_v - GG_u - FG_u}{2(EG - F^2)}, \quad \Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}.$$

- **Gauss equations and Codazzi-Mainardi equations**: Using the fact that $\mathbf{x}_{uuv} = \mathbf{x}_{uvu}$, $\mathbf{x}_{vvu} = \mathbf{x}_{vuv}$ and $\mathbf{n}_{uv} = \mathbf{n}_{vu}$, we easily get the following two equations which called the *Gauss equations*,

$$(\Gamma^2_{12})_u - (\Gamma^2_{11})_v + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12} = -EK;$$

$$(\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22} = FK;$$

and the following two equations which called the *Codazzi-Mainardi equations*:

$$e_v - f_u = e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11};$$

$$f_v - g_u = e\Gamma^1_{22} + f(\Gamma^2_{22} - \Gamma^1_{12}) - g\Gamma^2_{12}.$$

The Gauss equations allows us to calculate the Gauss curvature in terms of the first fundamental form only. Assume $F = 0$, then, from the Gauss euations above, we have

$$K = \frac{-1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right).$$

# 6 Vector Fields and Covariant Derivatives

- A vector field $\mathbf{w}$ on $S$ is a vector valued function, i.e. $\mathbf{w} : S \to \mathbf{R}^3$.

- Let $\mathbf{y} \in T_p(S)$. The *covariant derivative at* $p$ *of the vector field* $\mathbf{w}$ *relative to the vector* $\mathbf{y}$, denoted by $D_{\mathbf{y}}\mathbf{w}(p)$, is the tangential component of $\bigtriangledown_{\mathbf{y}}\mathbf{w}(p)$, where $\bigtriangledown_{\mathbf{y}}\mathbf{w}(p)$ is the directional derivative defined earlier.

- Let $\boldsymbol{\alpha} : I \to S$ be a parametrized curve in $S$ and let $\mathbf{w}$ be a vector field along $\boldsymbol{\alpha}$, then the *covariant derivative of the vector field* $\mathbf{w}$ *along the curve* $\boldsymbol{\alpha}$, denoted by $(D\mathbf{w}/dt)(t)$, is the tangential component of $(d\mathbf{w}/dt)(t)$.

  Let $\mathbf{x} : U \to \mathbf{R}^3$ be a parameterization for $S$. Since $\boldsymbol{\alpha}$ is a curve on $S$, we write $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$. Also, since $\mathbf{w}(t) \in T_{\boldsymbol{\alpha}(t)}S$, we can write

$$\mathbf{w}(t) = a(u(t), v(t))\mathbf{x}_u + b(u(t), v(t))\mathbf{x}_v = a(t)\mathbf{x}_u + b(t)\mathbf{x}_v.$$

  Thus, we have

$$\frac{d\mathbf{w}}{dt}(t) = a(\mathbf{x}_{uu}u' + \mathbf{x}_{uv}v') + b(\mathbf{x}_{vu}u' + \mathbf{x}_{vv}v') + a'\mathbf{x}_u + b'\mathbf{x}_v.$$

  By definition,

$$\frac{D\mathbf{w}}{dt} = pr \circ (d\mathbf{w}/dt),$$

  hence,

$$\frac{D\mathbf{w}}{dt} = (a' + \Gamma^1_{11}au' + \Gamma^1_{12}av' + \Gamma^1_{12}bu' + \Gamma^1_{22}bv')\mathbf{x}_u + (b' + \Gamma^2_{11}au' + \Gamma^2_{12}av' + \Gamma^2_{12}bu' + \Gamma^2_{22}bv')\mathbf{x}_v.$$

- **Remark**: $D\mathbf{w}/dt$ is an **intrinsic** geometric quantity whose expression in local coordinates involves Christoffel symbols.

# 7 Parallel Transport and Geodesics

- **Parallel Transport**: Let $\boldsymbol{\alpha} : I \to \mathbf{R}^3$ be a curve on a surface $S$. Let $\mathbf{w} : I \to \mathbf{R}^3$ be a tangential vector field along $\boldsymbol{\alpha}$. We say that $\mathbf{w}$ is parallel along $\boldsymbol{\alpha}$ if

$$D\mathbf{w}/dt = 0$$

  for every $t \in I$.

By the formular above, $\mathbf{w}$ *is parallel along* $\boldsymbol{\alpha}$ *if and only if, along* $\boldsymbol{\alpha}$*,*

$$a' + \Gamma_{11}^1 au' + \Gamma_{12}^1 av' + \Gamma_{12}^1 bu' + \Gamma_{22}^1 bv' = 0 \qquad (*)$$

*and*

$$b' + \Gamma_{11}^2 au' + \Gamma_{12}^2 av' + \Gamma_{12}^2 bu' + \Gamma_{22}^2 bv' = 0. \qquad (**)$$

Let $\boldsymbol{\alpha} : I \to \mathbf{R}^3$ be a curve on a surface $S$ and $\mathbf{w}_0 \in T_{\boldsymbol{\alpha}(t_0)}(S)$, $t_0 \in I$. The vector $\mathbf{w}_1 \in T_{\boldsymbol{\alpha}(t_1)}(S)$, $t_1 \in I$ is said to be the Parallel Transport of $\mathbf{w}_0$ along $\boldsymbol{\alpha}$ if there exists a parallel vector field $\mathbf{w}(t)$ along $\boldsymbol{\alpha}$ such that $\mathbf{w}_0 = \mathbf{w}(t_0)$ and $\mathbf{w}_1 = \mathbf{w}(t_1)$.

- **Geodesics, and geodesic equations**: Let $\boldsymbol{\alpha} : I \to S$ be a parametrized curve on a surface $S$. Then $\boldsymbol{\alpha}$ is *a geodesic* if and only if the field of $\boldsymbol{\alpha}'(t)$ is parallel along $\boldsymbol{\alpha}$, i.e.,

$$\frac{D\boldsymbol{\alpha}'(t)}{dt} = 0,$$

on $I$.

A regular connected curve $C$ in $S$ is siad to be a *geodesic* if, for every $p \in C$, the parametrization $\boldsymbol{\alpha}(s)$ of a coordinate neighborhood of $p$ by the **arc length** $s$ is geodesic, i.e., field of $\boldsymbol{\alpha}'(s)$ is parallel along $\boldsymbol{\alpha}$.

Let $\mathbf{x} : U \to S$ be a parametrization for $S$, and let $\boldsymbol{\alpha}$ be a curve on $S$. Write $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$. Then $\boldsymbol{\alpha}$ is a geodesic if and only if $(D\boldsymbol{\alpha}'(t)/dt) = 0$, that is

$$u'' + \Gamma_{11}^1 (u')^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 (v')^2 = 0 \qquad (*)$$

and

$$v'' + \Gamma_{11}^2 (u')^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 (v')^2 = 0. \qquad (**)$$

- **Geodesic Curvature:** Recall that

$$\kappa_g = \boldsymbol{\alpha}'' \cdot (\mathbf{n} \times \boldsymbol{\alpha}').$$

It is easy to check that *a curve* $\boldsymbol{\alpha}$ *in the surface* $S$ *is geodesic if and only if* $\kappa_g \equiv 0$.

- **Liouville's theorem**:

**Liouville's theorem**: *Let* $\mathbf{x}(u, v)$ *be an* **orthgonal** *parametrization (i.e.,* $F = 0$*), then*

$$\kappa_g = \frac{1}{2\sqrt{EG}} \left\{ G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right\} + \frac{d\phi}{ds},$$

*where* $\phi(s)$ *is the angle that* $\mathbf{x}_u$ *makes with* $\boldsymbol{\alpha}'(s)$ *in the given orientation. In particular, we can write:*

$$\kappa_g = (\kappa_g)_1 \cos\phi + (\kappa_g)_2 \sin\phi + \frac{d\phi}{dt},$$

*where* $(\kappa_g)_1$ *and* $(\kappa_g)_2$ *are the geodesic curvatures of the coordinate curves* $v = cost$ *and* $u = cost$ *respectively.*

Let $\phi_{12}(s) := \frac{1}{2\sqrt{EG}}(G_u v' - E_v u')$, then we can write

$$\kappa_g = \phi_{12}(s) + \phi'(s).$$

Here we give a direct proof of above formula(Liouville's formula). Let

$$\mathbf{e}_1 = \frac{\mathbf{x}_u}{\sqrt{E}}, \mathbf{e}_2 = \frac{\mathbf{x}_v}{\sqrt{G}},$$

then $\mathbf{e}_1, \mathbf{e}_2$ gives an orthonomal basis for $T_p(S)$. Write $\boldsymbol{\alpha}(s) = \mathbf{x}(u(s), v(s))$, and we set

$$\phi_{12} = \frac{d}{ds} < \mathbf{e}_1(u(s), v(s)), \mathbf{e}_2(u(s), v(s)) >,$$

which we may write more cacually as $\phi_{12} = \mathbf{e}_1'(s) \cdot \mathbf{e}_2(s)$. Then (take the full advantage of the orthogonality of $\mathbf{x}_u$ and $\mathbf{x}_v$),

$$\begin{aligned}
\phi_{12} &= \left(\frac{d}{ds}\left(\frac{\mathbf{x}_u}{\sqrt{E}}\right)\right) \cdot \left(\frac{\mathbf{x}_v}{\sqrt{G}}\right) \\
&= \frac{1}{\sqrt{EG}}(\mathbf{x}_{uu}u' + \mathbf{x}_{uv}v') \cdot \mathbf{x}_v \\
&= \frac{1}{\sqrt{EG}}((\Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v)u' + (\Gamma^1_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v)v') \cdot \mathbf{x}_v \\
&= \frac{G}{\sqrt{EG}}(\Gamma^2_{11}u' + \Gamma^2_{12}v') = \frac{1}{2\sqrt{EG}}(G_u v' - E_v u').
\end{aligned}$$

We now show that $\kappa_g = \phi_{12}(s) + \phi'(s)$. In fact, $\kappa_g = \boldsymbol{\alpha}'' \cdot (\mathbf{n} \times \boldsymbol{\alpha}')$. Now, since $\boldsymbol{\alpha}' = \cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2$, $\mathbf{n} \times \boldsymbol{\alpha}' = -\sin\theta \mathbf{e}_1 + \cos\theta \mathbf{e}_2$. Hence, by a calculation, we have $\kappa_g = \phi_{12} + \theta'$. This proves the formula.

Note, the above formula and the formula for Gauss curvature (assume $F = 0$)

$$K = \frac{-1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$$

are the key to prove Gauss-Bennet (in applying the Green's formula).

- **Angle change of the parallel vector field along the curve $\boldsymbol{\alpha}$**: Let $\boldsymbol{\alpha} : [0, l] \to S$ be a closed curve in $S$. Let $C$ be the tarce of $\boldsymbol{\alpha}$. Let $\mathbf{w}(t)$ to be the parallel transprot of $\mathbf{v}_0 \in T_{\boldsymbol{\alpha}(0)}S$ along C, write $\mathbf{w}(t) = \cos\psi(t)\mathbf{e}_1 + \sin\psi(t)\mathbf{e}_2$, taking $\psi(0) = 0$. Then $\mathbf{w}$ is parallel along $\boldsymbol{\alpha}$ if and only if $\phi_{12} + \psi' = 0$. Hence we have

$$\triangle\psi = \psi(l) - \psi(0) = -\int_0^l \phi_{12}(s)ds.$$

On the other hand, by the Gauss curvature formula above and by Green's theorem, we have

$$\int_0^l \phi_{12}(s)ds = -\int\int_{int(\boldsymbol{\alpha})} K d\sigma,$$

where $int(\boldsymbol{\alpha})$ means the interior of the curve $\boldsymbol{\alpha}$. Hence

$$\triangle\psi = \psi(l) - \psi(0) = \int\int_{int(\boldsymbol{\alpha})} K d\sigma.$$

# 8 The Gauss-Bonnet Theorem

**Gauss-Bonnet Theorem(Local)**. Let $\mathbf{x} : U \to S$ be an orthgonal parametrization (i.e $F = 0$) of a neighborhood of an oriented surface $S$, where $U \subset \mathbf{R}^2$ is homeomorphic to an open disk. Let $R \subset \mathbf{x}(U)$ be a simple region of $S$ and let $\boldsymbol{\alpha} : I \to S$ be such that $\partial R = \boldsymbol{\alpha}(I)$. Assume that $\boldsymbol{\alpha}$ is positively oriented, parametrized by arc length $s$, and let $\boldsymbol{\alpha}(s_0), \ldots, \boldsymbol{\alpha}(s_k)$ and $\theta_0, \ldots, \theta_k$ be, respectively, the vertices and the external angles of $\boldsymbol{\alpha}$. Then

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \kappa_g(s) ds + \int\int_R K d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi,$$

or we can write

$$\int_{\partial R} \kappa_g(s) ds + \int\int_R K d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi,$$

**Gauss-Bonnet Theorem**. Let $R \subset S$ be a regular region of an oriented surface and let $C_1, \ldots, C - n$ be the closed, simple, piecewise regular curves which from $\partial R$. Suposed that $C_i$ is positively oriented and let $\theta_1, \ldots, \theta_p$ be the set of external angles of $C_1, \ldots, C_n$. Then

$$\int_{\partial R} \kappa_g(s) ds + \int\int_R K d\sigma + \sum_{i=1}^{p} \theta_i = 2\pi\chi(R),$$

where $\chi(R)$ is he Euler-Poincare characteristic of $R$.

In particular, if $S$ is an orientable **compact** surface, then

$$\int\int_S K d\sigma = 2\pi\chi(S).$$