

VECTORS and POLAR COORDINATES

LEARNING OBJECTIVES

In this section, you will:

- View vectors geometrically.
- Find magnitude and direction.
- Perform vector addition and scalar multiplication.
- Find the component form of a vector.
- Find the unit vector in the direction of v .
- Perform operations with vectors in terms of i and j .
- Find the dot product of two vectors.
- Identify orthogonal vectors.

9.1 VECTORS

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140° . A north wind (from north to south) is blowing at 16.2 miles per hour, as shown in **Figure 1**. What are the ground speed and actual bearing of the plane?

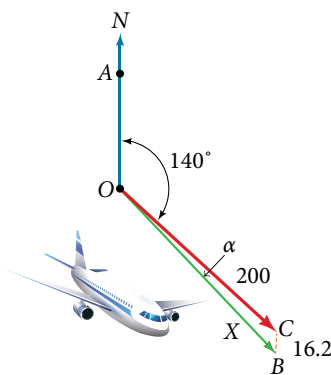


Figure 1

Ground speed refers to the speed of a plane relative to the ground. Airspeed refers to the speed a plane can travel relative to its surrounding air mass. These two quantities are not the same because of the effect of wind. Later in this section, we will find the airplane's groundspeed and bearing, while investigating another approach to problems of this type. First, however, let's examine the basics of vectors.

A Geometric View of Vectors

A **vector** is a specific quantity drawn as a line segment with an arrowhead at one end. It has an **initial point**, where it begins, and a **terminal point**, where it ends. A vector is defined by its **magnitude**, or the **length of the line**, and its **direction**, indicated by an arrowhead at the terminal point. Thus, a vector is a directed line segment. There are various symbols that distinguish vectors from other quantities:

- **Lower case**, boldfaced type, with or without an arrow on top such as \mathbf{v} , \mathbf{u} , \mathbf{w} , \vec{v} , \vec{u} , \vec{w} .
- Given initial **point P** and **terminal point Q**, a vector can be represented as \overrightarrow{PQ} . The arrowhead on top is what indicates that it is not just a line, but a directed line segment.
- Given an **initial point of (0, 0)** and **terminal point (a, b)**, a vector may be represented as $\langle a, b \rangle$.

This last symbol $\langle a, b \rangle$ has special significance. It is called the **standard position**. The position vector has an initial point (0, 0) and a terminal point (a, b) . To change any vector into the position vector, we think about the **change in the x-coordinates** and the change in the **y-coordinates**. Thus, if the initial point of a vector \overrightarrow{CD} is $C(x_1, y_1)$ and the terminal point is $D(x_2, y_2)$, then the position vector is found by calculating

$$\overrightarrow{CD} = \langle x_2 - x_1, y_2 - y_1 \rangle = \langle a, b \rangle$$

Δx Δy
 a b

In Figure 2, we see the original vector \vec{CD} and the position vector \vec{AB} .

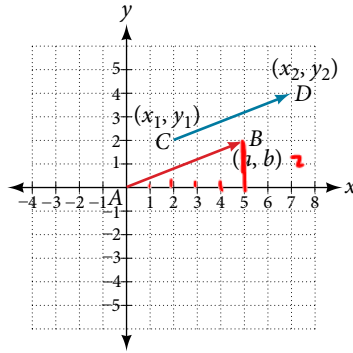


Figure 2

$$C: (2, 2) \quad D: (7, 4)$$

$$\vec{CD} = \langle 7-2, 4-2 \rangle$$

$$= \langle 5, 2 \rangle$$

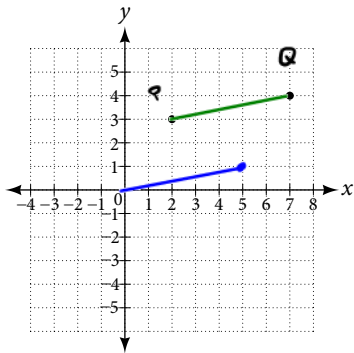
$$\vec{AB} = \langle 5, 2 \rangle$$

Properties of Vectors

- i) A vector is a directed line segment with an initial point and a terminal point.
- ii) Vectors are identified by magnitude i.e. the length of the line, and by direction represented by the arrowhead pointing toward the terminal point.
- iii) The position vector $\langle a, b \rangle$ has an initial point at $(0, 0)$ and is identified by its terminal point (a, b) .

Example 1 Consider the vector whose initial point is $P(2, 3)$ and terminal point is $Q(7, 4)$. Find the position vector.

Solution



$$\vec{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle = \langle 7-2, 4-3 \rangle$$

from (0,0)

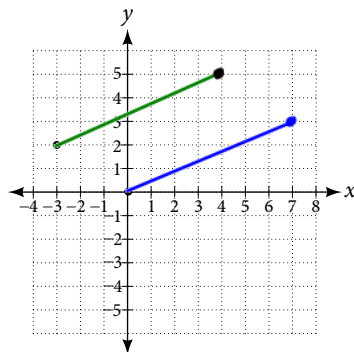
$$= \langle 5, 1 \rangle = \langle a, b \rangle$$

$$\vec{u} = \langle 5, 1 \rangle$$

Example 2 Find the position vector given that vector v has an initial point at $(-3, 2)$ and a terminal point at $(4, 5)$,

then graph both vectors in the same plane.

Solution



$$\vec{v} = \langle x_2 - x_1, y_2 - y_1 \rangle = \langle 4 - (-3), 5 - 2 \rangle$$

$$= \langle 7, 3 \rangle$$

$$\vec{u} = \langle 7, 3 \rangle$$

Finding Magnitude and Direction

To work with a vector, we need to be able to find its magnitude and its direction. We find its magnitude using the Pythagorean Theorem or the distance formula, and we find its direction using the inverse tangent function.

Magnitude and Direction of a Vector

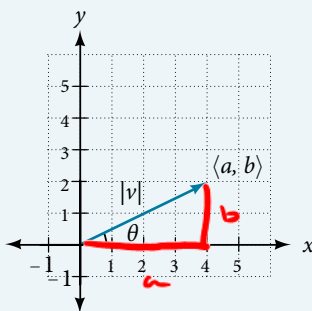
Given a position vector $v = \langle a, b \rangle$, the magnitude is found by $|v| = \sqrt{a^2 + b^2}$.

The direction is equal to the angle formed with the x -axis, or with the y -axis, depending on the application.

For a position vector, the direction is found by $\tan \theta = \frac{b}{a} \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$, as illustrated in **Figure 5**.

$$\tan \theta = \frac{y}{x} = \frac{b}{a}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$



$|v| = \text{Pyth. Thm.}$

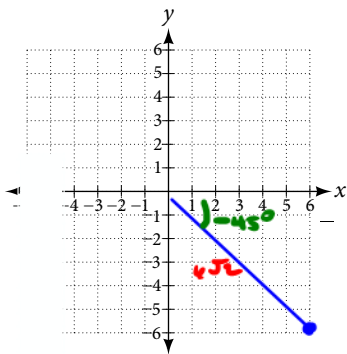
Figure 5

Two vectors v and u are considered equal if they have the same magnitude and the same direction.

Additionally, if both vectors have the same position vector, they are equal.

Example 3 Find the magnitude and direction of the vector with initial point $P(-8, 1)$ and terminal point $Q(-2, -5)$. Draw the vector.

Solution



$$\vec{PQ} = \langle -2 - (-8), -5 - 1 \rangle$$

$$= \langle 6, -6 \rangle$$

$$\vec{u} = \langle 6, -6 \rangle$$

$$\tan \theta = \frac{y}{x} = \frac{-6}{6} = -1$$

$$\theta = \tan^{-1}(-1)$$

$$= -\frac{\pi}{4} \text{ or } -45^\circ$$

$$|PQ| = \sqrt{(6)^2 + (-6)^2}$$

$$= \sqrt{36 + 36}$$

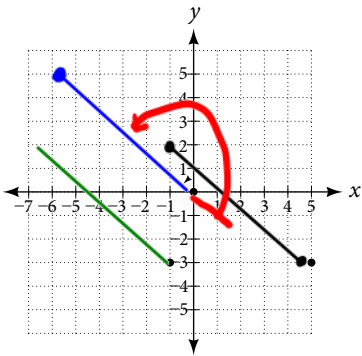
$$= \sqrt{72}$$

$$= 6\sqrt{2}$$

Example 4 Show that vector v with initial point at $(5, -3)$ and terminal point at $(-1, 2)$ is equal to vector u with initial point at $(-1, -3)$ and terminal point at $(-7, 2)$.

Draw the position vector on the same grid as v and u . Next, find the magnitude and direction of each vector.

Solution



$$\vec{v} = \langle -1 - 5, 2 - (-3) \rangle = \langle -6, 5 \rangle \quad \vec{u} = \langle -7 - (-1), 2 - (-3) \rangle = \langle -6, 5 \rangle$$

$$\vec{w} = \langle -6, 5 \rangle \text{ position vector}$$

$$|\vec{v}| = \sqrt{(-6)^2 + (5)^2} = \sqrt{36 + 25} = \sqrt{61}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \tan^{-1}\left(\frac{5}{-6}\right) \approx -39.81^\circ + 180.00$$

$$\theta = 140.19^\circ$$

Performing Vector Addition and Scalar Multiplication

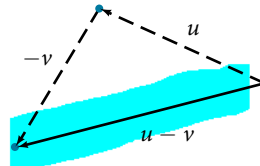
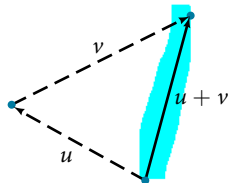
The sum of two vectors u and v , or **vector addition**, produces a third vector $u + v$, the **resultant** vector.

To find $u + v$, we have the initial point of v meet the terminal end of u .

This position corresponds to the notion that we move along the first vector and then, from its terminal point, we move along the second vector.

The sum $u + v$ is the resultant vector because it results from addition or subtraction of two vectors.

The resultant vector travels directly from the beginning of u to the end of v in a straight path.

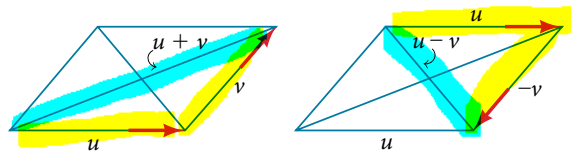


Vector subtraction is similar to vector addition. To find $u - v$, view it as $u + (-v)$.

Adding $-v$ is reversing direction of v and adding it to the end of u .

The new vector begins at the start of u and stops at the end point of $-v$.

See **Figure below** for a visual that compares vector addition and vector subtraction *using parallelograms*.



Vector Addition / Subtraction

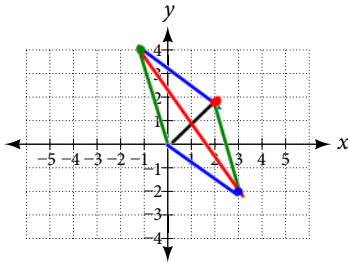
Given two vectors $u = \langle a, b \rangle$ and $v = \langle c, d \rangle$, then

the sum of these two vectors is $u + v = \langle a + c, b + d \rangle$, and

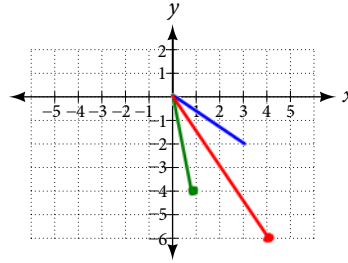
the difference of these two vectors is $u - v = u + (-v) = \langle a - c, b - d \rangle$

Example 5 Given $u = \langle 3, -2 \rangle$ and $v = \langle -1, 4 \rangle$, find two new vectors $u + v$, and $u - v$.

Solution



$$\begin{aligned} \vec{u} + \vec{v} &= \langle 3 + (-1), -2 + 4 \rangle \\ &= \langle 2, 2 \rangle \end{aligned}$$



$$\begin{aligned} \vec{u} - \vec{v} &= \langle 3 - (-1), -2 - 4 \rangle \\ &= \langle 4, -6 \rangle \end{aligned}$$

$$\begin{aligned} -\vec{v} &= -\langle -1, 4 \rangle \\ &= \langle 1, -4 \rangle \end{aligned}$$

Multiplying By a Scalar

Adding and Subtracting vectors gives a new vector with a different magnitude and direction.

The process of multiplying a vector by a scalar/constant, changes only the magnitude of the vector.

Scalar multiplication has no effect on the direction unless the scalar is negative,

in which case the direction of the resulting vector is opposite the direction of the original vector.

Scalar Multiplication

Scalar multiplication involves the product of a vector and a scalar.

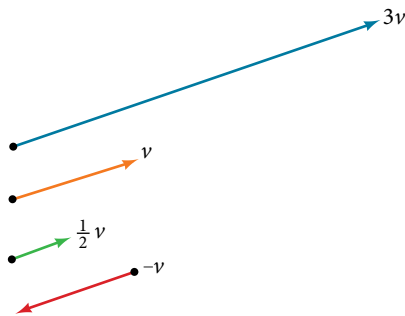
Each component of the vector is multiplied by the scalar. Thus, to multiply $\vec{v} = \langle a, b \rangle$ by k , we have

$$k\vec{v} = \langle ka, kb \rangle$$

Only the magnitude changes, unless k is negative, and then the vector reverses direction.

Example 6 Given vector $v = \langle 3, 1 \rangle$, find $3v$, $\frac{1}{2}v$, and $-v$.

Solution



$$3\vec{v} = 3\langle 3, 1 \rangle = \langle 9, 3 \rangle$$

$$\frac{1}{2}\vec{v} = \frac{1}{2}\langle 3, 1 \rangle = \langle \frac{3}{2}, \frac{1}{2} \rangle$$

$$-\vec{v} = -1 \cdot \langle 3, 1 \rangle = \langle -3, -1 \rangle$$

Example 7 Given $u = \langle 3, -2 \rangle$ and $v = \langle -1, 4 \rangle$, find a new vector $w = 3u + 2v$.

Solution

$$\begin{aligned}\vec{w} &= 3\vec{u} + 2\vec{v} \\ &= 3\langle 3, -2 \rangle + 2\langle -1, 4 \rangle \\ &= \langle 9, -6 \rangle + \langle -2, 8 \rangle \\ &= \langle 7, 2 \rangle\end{aligned}$$

Finding Component Form

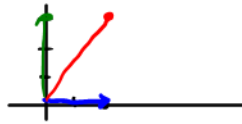
In some applications, it is helpful for us to be able to break a vector down into its components.

Vectors are comprised of two components:

the horizontal component is the x direction, and the vertical component is the y direction.

For example, think of the position vector $\langle 2, 3 \rangle$ as a sum of the vectors $v_1 = \langle 2, 0 \rangle$ and $v_2 = \langle 0, 3 \rangle$.

$$v = \langle 2 + 0, 3 + 0 \rangle = \langle 2, 3 \rangle$$



Using the Pythagorean Theorem, the magnitude of v_1 is 2, and the magnitude of v_2 is 3.

To find the magnitude of v , use the formula with the position vector.

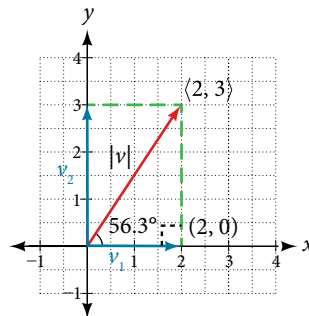
$$\begin{aligned}|v| &= \sqrt{|v_1|^2 + |v_2|^2} \\ &= \sqrt{2^2 + 3^2} \\ &= \sqrt{13}\end{aligned}$$

The magnitude of v is $\sqrt{13}$. To find the direction, we use the tangent function $\tan \theta = \frac{y}{x}$.

$$\tan \theta = \frac{v_2}{v_1}$$

$$\tan \theta = \frac{3}{2}$$

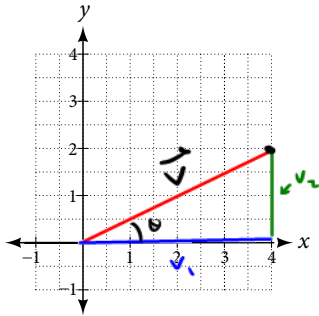
$$\theta = \tan^{-1}\left(\frac{3}{2}\right) = 56.3^\circ$$



Thus, the magnitude of v is $\sqrt{13}$ and the direction is 56.3° off the horizontal.

Example 8 Find the components of the vector v with initial point $(3, 2)$ and terminal point $(7, 4)$.

Solution



$$\begin{aligned}\vec{v} &= \langle 7-3, 4-2 \rangle \\ &= \langle 4, 2 \rangle\end{aligned}$$

$$\begin{aligned}\vec{v} &= v_1 + v_2 \\ &= \langle 4, 0 \rangle + \langle 0, 2 \rangle \\ &= \langle 4, 2 \rangle\end{aligned}$$

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{y}{x}\right) \\ &= \tan^{-1}\left(\frac{2}{4}\right) \\ &= 26.57^\circ\end{aligned}$$

Finding the Unit Vector in the Direction of v

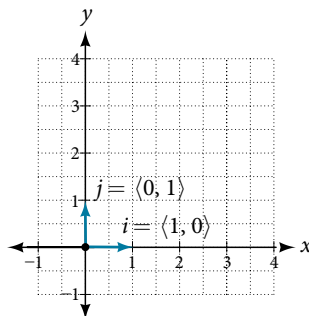
In addition to finding a vector's components, it is also useful in solving problems to find a vector in the same direction as the given vector, but of magnitude 1. We call a vector with a magnitude of 1 a **unit vector**.

We can then preserve the direction of the original vector while simplifying calculations.

Unit vectors are defined in terms of components:

The **horizontal unit vector** is written as $i = \langle 1, 0 \rangle$ and is directed along the positive horizontal axis.

The **vertical unit vector** is written as $j = \langle 0, 1 \rangle$ and is directed along the positive vertical axis.

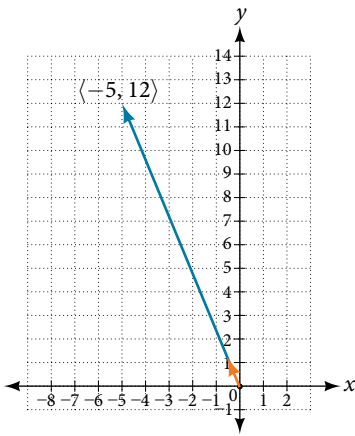


The Unit Vectors

If v is a nonzero vector, then $\frac{v}{|v|}$ is a unit vector in the direction of v . Any vector divided by its magnitude is a unit vector. Notice that **magnitude** is always a **scalar**, and **dividing by a scalar** is the same as multiplying by the reciprocal of the scalar.

Example 9 Find a unit vector in the same direction as $v = \langle -5, 12 \rangle$.

Solution



$$\begin{aligned} |v| &= \sqrt{(-5)^2 + (12)^2} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

$$\begin{aligned} \frac{v}{|v|} &= \frac{\langle -5, 12 \rangle}{13} = \frac{1}{13} \langle -5, 12 \rangle \\ &= \left\langle \frac{-5}{13}, \frac{12}{13} \right\rangle = \frac{-5}{13} i + \frac{12}{13} j \end{aligned}$$

Performing Operations with Vectors in Terms of i and j

So far, we have investigated the basics of vectors:

magnitude and direction, vector addition and subtraction, scalar multiplication, the components of vectors, and the representation of vectors geometrically.

Now, we will represent vectors in rectangular coordinates in terms of i and j .

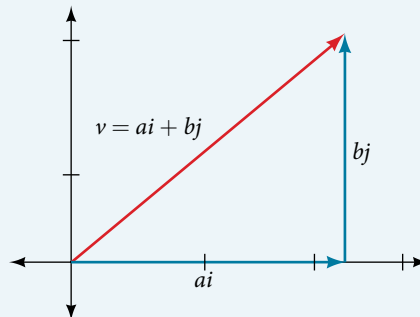
Vectors in the Rectangular Plane

Given a vector v with initial point $P = (x_1, y_1)$ and terminal point $Q = (x_2, y_2)$, v is written as

$$v = (x_2 - x_1)i + (y_2 - y_1)j$$

The position vector from $(0, 0)$ to (a, b) , where $(x_2 - x_1) = a$ and $(y_2 - y_1) = b$, is written as $v = ai + bj$. This vector sum is called a linear combination of the vectors i and j .

The magnitude of $v = ai + bj$ is given as $|v| = \sqrt{a^2 + b^2}$. See **Figure 16**.



Example 10 Given a vector v with initial point $P = (2, -6)$ and terminal point $Q = (-6, 6)$, write the vector in terms of i and j .

Solution

$$\begin{aligned}\vec{v} &= \langle -6-2, 6-(-6) \rangle \\ &= \langle -8, 12 \rangle\end{aligned}$$

$$\vec{v} = -8i + 12j$$

Example 11 Given initial point $P_1 = (-1, 3)$ and terminal point $P_2 = (2, 7)$, write the vector v in terms of i and j .

Solution

$$\begin{aligned}\overrightarrow{P_1 P_2} = \vec{v} &= \langle 2-(-1), 7-3 \rangle \\ &= \langle 3, 4 \rangle\end{aligned}$$

$$\vec{v} = 3i + 4j$$

Performing Operations on Vectors in Terms of i and j

When vectors are written in terms of i and j , we can carry out addition, subtraction, and scalar multiplication by performing operations on corresponding components.

Adding and Subtracting vectors in rectangular Coordinates

Given $v = ai + bj$ and $u = ci + dj$, then

$$v + u = (a + c)i + (b + d)j$$

$$v - u = (a - c)i + (b - d)j$$

Example 12 Find the sum of $v_1 = 2i - 3j$ and $v_2 = 4i + 5j$.

Solution

$$v_1 + v_2$$
$$\begin{array}{r} 2i - 3j + 4i + 5j \\ \hline \end{array}$$

$$6i + 2j$$

Calculating the Component Form of a Vector: Direction

We have seen how to draw vectors according to their initial and terminal points and how to find the position vector. We have also examined notation for vectors drawn specifically in the Cartesian coordinate plane using i and j . For any of these vectors, we can calculate the magnitude.

Now, we want to combine the key points, and look further at the ideas of magnitude and direction.

Calculating direction follows the same straightforward process we used for polar coordinates.

We find the direction of the vector by finding the angle to the horizontal.

We do this by using the basic trigonometric identities, but with $|v|$ replacing r .

Vector Components in terms of Magnitude and Direction

Given a position vector $v = \langle x, y \rangle$ and a direction angle θ ,

$$\cos \theta = \frac{x}{|v|} \quad \text{and} \quad \sin \theta = \frac{y}{|v|}$$

$$x = |v| \cos \theta \quad y = |v| \sin \theta$$

Thus, $v = xi + yj = |v|\cos \theta i + |v|\sin \theta j$, and magnitude is expressed as $|v| = \sqrt{x^2 + y^2}$.

Example 13 Write a vector with length 7 at an angle of 135° to the positive x -axis in terms of magnitude and direction.

Solution

$$\begin{aligned} x &= |v| \cdot \cos \theta & y &= |v| \cdot \sin \theta \\ x &= 7 \cdot \cos(135^\circ) & &= 7 \cdot \sin(135^\circ) \\ x &= 7 \cdot \left(-\frac{\sqrt{2}}{2}\right) & &= 7 \cdot \frac{\sqrt{2}}{2} \\ &= -\frac{7\sqrt{2}}{2} & &= \frac{7\sqrt{2}}{2} \end{aligned}$$

$$\vec{v} = -\frac{7\sqrt{2}}{2} i + \frac{7\sqrt{2}}{2} j$$

Finding the Dot Product of Two Vectors

Scalar multiplication involves multiplying a vector by a scalar, and the result is a vector.

As we have seen, multiplying a vector by a number is called scalar multiplication.

If we multiply a vector by a vector, there are two possibilities: the **dot product** and the **cross product**. We will only examine the dot product here; you may encounter the cross product in more advanced mathematics courses.

The **dot product of two vectors** involves multiplying two vectors together, and the **result is a scalar**.

Dot Product

The **dot product** of two vectors $v = \langle a, b \rangle$ and $u = \langle c, d \rangle$ is the sum of the product of the horizontal components and the product of the vertical components.

$$v \cdot u = ac + bd$$

To find the angle between the two vectors, use the formula below.

$$\cos \theta = \frac{v \cdot u}{|v| |u|}$$

Example 14 Find the dot product of $v = \langle 5, 12 \rangle$ and $u = \langle -3, 4 \rangle$.

Solution $\vec{v} \cdot \vec{u} = 5(-3) + 12(4)$

$$= -15 + 48 = 33$$

Example 15 Find the dot product of $v_1 = 5i + 2j$ and $v_2 = 3i + 7j$. Then, find the angle between the two vectors.

Solution

$$\begin{aligned} \vec{v}_1 \cdot \vec{v}_2 &= 5 \cdot 3 + 2(7) \\ &= 15 + 14 \\ &= 29 \end{aligned}$$

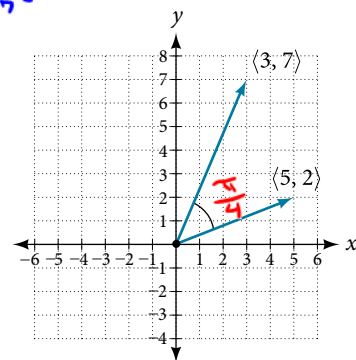
$$\begin{aligned} |v_1| &= \sqrt{5^2 + 2^2} \\ &= \sqrt{25 + 4} \\ &= \sqrt{29} \end{aligned}$$

$$\begin{aligned} |v_2| &= \sqrt{3^2 + 7^2} \\ &= \sqrt{9 + 49} \\ &= \sqrt{58} \end{aligned}$$

$$\cos \theta = \frac{v_1 \cdot v_2}{|v_1| |v_2|} = \frac{29}{\sqrt{29} \cdot \sqrt{58}} = \frac{\sqrt{29}}{\sqrt{58}} = \sqrt{\frac{29}{58}} = \sqrt{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\theta = \frac{\pi}{4} \text{ or } 45^\circ$$



Example 16 Find the angle between $u = \langle -3, 4 \rangle$ and $v = \langle 5, 12 \rangle$.

Solution

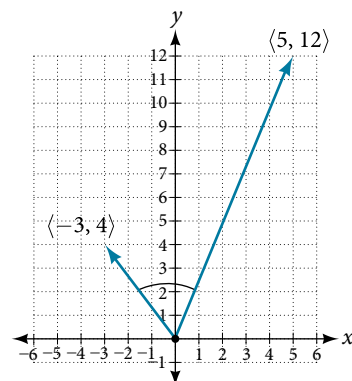
$$\begin{aligned} |u| &= \sqrt{9 + 16} \\ &= \sqrt{25} = 5 \end{aligned}$$

$$\begin{aligned} |v| &= \sqrt{25 + 144} \\ &= \sqrt{169} = 13 \end{aligned}$$

$$\vec{u} \cdot \vec{v} = (-3)5 + 4(12) = -15 + 48 = 33$$

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|u| |v|} = \frac{33}{5 \cdot 13} = \frac{33}{65}$$

$$\cos \theta = \frac{33}{65} \Rightarrow \theta = \cos^{-1}\left(\frac{33}{65}\right) = 59.5^\circ$$



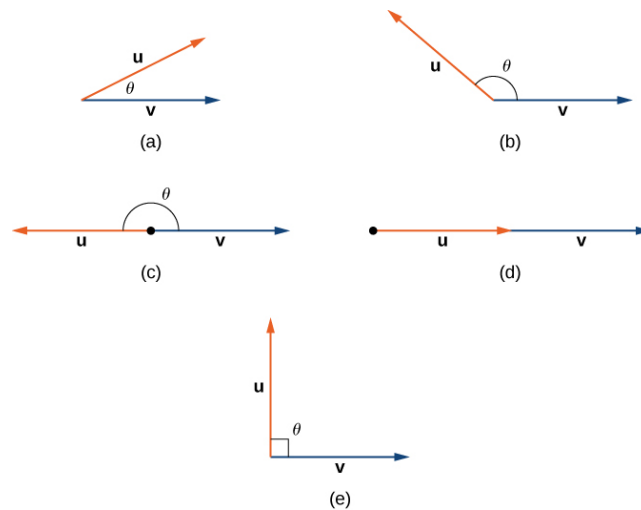
Identifying Orthogonal Vectors

The dot product of two vectors involves multiplying two vectors together, and the result is a scalar. The dot product serves as a tool to measure the angle formed by the pair of vectors involved.

The angle formed can be acute ($0 < \cos \theta < 1$), obtuse ($-1 < \cos \theta < 0$), or straight ($\cos \theta = -1$).

If $\cos \theta = 1$, then both vectors have same direction.

If $\cos \theta = 0$, then the vectors when placed in standard position, form a right angle.



Orthogonal Vectors

Two given nonzero vectors $v = \langle a, b \rangle$ and $u = \langle c, d \rangle$ are called **orthogonal vectors**

if and only if their dot product is zero, i.e. $v \cdot u = 0$.

The terms *orthogonal*, *perpendicular*, and *normal* indicate that mathematical objects are intersecting at right angles. The use of each term is determined mainly by its context: We say that vectors are orthogonal and lines are perpendicular. The term *normal* is used mostly when measuring the angle made with a plane or other surface.

Example 17 Determine whether the vectors $v = \langle 5, 4 \rangle$ and $u = \langle -4, 5 \rangle$ are orthogonal or not.

Solution $\vec{u} \cdot \vec{v} = 5(-4) + 4(5)$
 $= -20 + 20$
 $= 0 \rightarrow$ Orthogonal vectors

Example 18 Find the value of x for which the vectors $v = \langle 2x, 8 \rangle$ and $u = \langle 4, 3 \rangle$ are orthogonal.

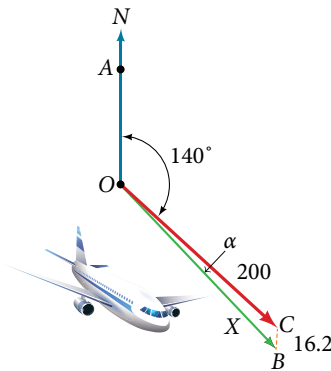
Solution $\vec{v} \cdot \vec{u} = 2x(4) + 8(3)$
 $= 8x + 24$
 $8x + 24 = 0$
 $x = -3$

$u \cdot v = 0$

Example 19 Finding Ground Speed and Bearing Using Vectors

We now have the tools to solve the problem we introduced in the opening of the section.

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140° . A north wind (from north to south) is blowing at 16.2 miles per hour. What are the ground speed and actual bearing of the plane?



Solution The ground speed is represented by x in the diagram, and we need to find the angle α in order to calculate the adjusted bearing, which will be $140^\circ + \alpha$.

Notice in **Figure above**, that angle BCO must be equal to angle AOC by the rule of alternating interior angles, so angle BCO is 140° . We can find x by the **Law of Cosines**:

$$x^2 = (16.2)^2 + (200)^2 - 2(16.2)(200)\cos(140^\circ)$$

$$x^2 = 45,226.41$$

$$x = \sqrt{45,226.41}$$

$$x = 212.7$$

The ground speed is approximately 213 miles per hour. Now we can calculate the bearing using the Law of Sines.

$$\frac{\sin \alpha}{16.2} = \frac{\sin(140^\circ)}{212.7}$$

$$\sin \alpha = \frac{16.2\sin(140^\circ)}{212.7}$$

$$= 0.04896$$

$$\sin^{-1}(0.04896) = 2.8^\circ$$

Therefore, the plane has a SE bearing of $140^\circ + 2.8^\circ = 142.8^\circ$. The ground speed is 212.7 miles per hour.

Paper 23

15 A

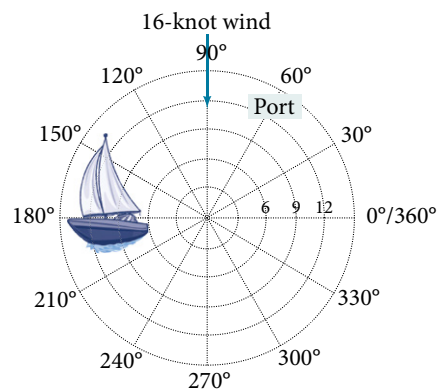
LEARNING OBJECTIVES

In this section, you will:

- Plot points using polar coordinates.
- Convert from polar coordinates to rectangular coordinates.
- Convert from rectangular coordinates to polar coordinates.
- Transform equations between polar and rectangular forms.
- Identify and graph polar equations by converting to rectangular equations.

9.2 POLAR COORDINATES

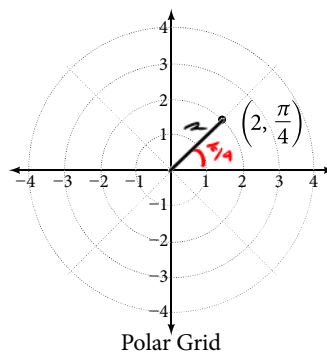
Over 12 kilometers from port, a sailboat encounters rough weather and is blown off course by a 16-knot wind. How can the sailor indicate his location to the Coast Guard? In this section, we will investigate a method of representing location that is different from a standard coordinate grid.



Plotting Points Using Polar Coordinates

When we think about plotting points in the plane, we usually think of **rectangular coordinates (x, y)** in the Cartesian coordinate plane. However, there are other ways of writing a coordinate pair and other types of grid systems. In this section, we introduce to **polar coordinates**, which are points labeled **(r, θ)** and plotted on a polar grid. The polar grid is represented as a series of concentric circles radiating out from the **pole**, or the origin of the coordinate plane.

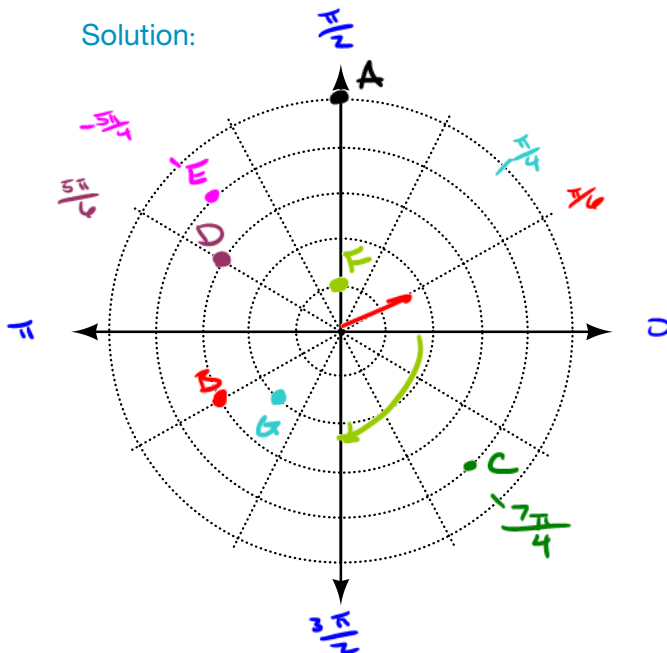
The polar grid is scaled as the unit circle with the **positive x -axis** now viewed as the **polar axis** and the **origin as the pole**. The first coordinate r is the radius or length of the directed line segment from the pole. The angle θ , measured in radians, indicates the direction of r . We move counterclockwise from the polar axis by an angle of θ , and measure a directed line segment the length of r in the direction of θ . Even though we measure θ first and then r , the polar point is written with the r -coordinate first. For example, to plot the point $(2, \frac{\pi}{4})$, we would move $\frac{\pi}{4}$ units in the **counterclockwise direction** and then a **length of 2** from the pole. This point is plotted on the grid in **Figure below**



Example 1 Plot the following points on the polar grid.

- Distance → Angle ↻
- $(5, \frac{\pi}{2}) \rightarrow A$
 - $(-3, \frac{\pi}{6}) \rightarrow B$
 - $(4, \frac{7\pi}{4}) \rightarrow C$
 - $(3, \frac{5\pi}{6}) \rightarrow D$
 - $(4, \frac{-5\pi}{4}) \rightarrow E$
 - $(-1, -\frac{\pi}{2}) \rightarrow F$
 - $(-2, \frac{\pi}{4}) \rightarrow G$

Solution:



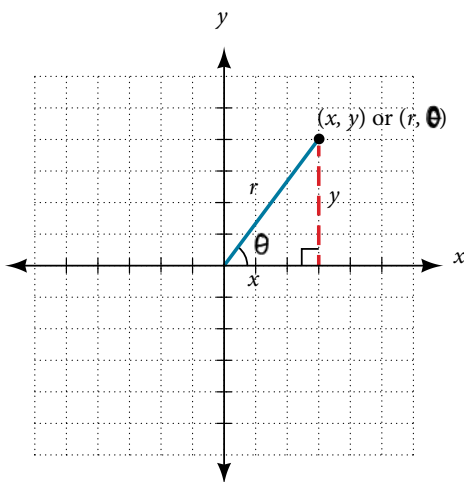
Converting from Polar Coordinates to Rectangular Coordinates

When given a set of polar coordinates, we may need to convert them to rectangular coordinates.

To do so, we can recall the relationships that exist among the variables x , y , r , and θ .

Dropping a perpendicular from the point in the plane to the x -axis forms a right triangle, as illustrated **below**.

In this right triangle, we define $\cos \theta$ and $\sin \theta$ using the sides with length x and y , and the hypotenuse r .



$$\frac{x}{r} = \cos \theta$$

$$\frac{y}{r} = \sin \theta$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

Converting from Polar Coordinates to Rectangular Coordinates

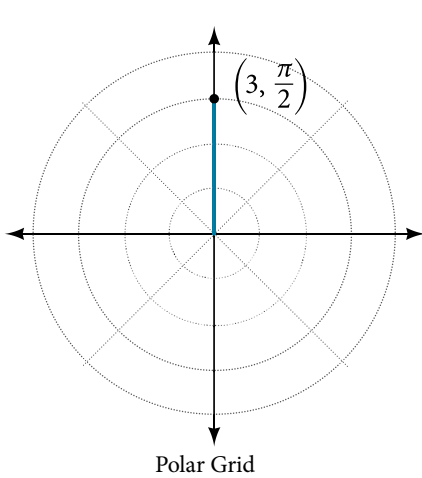
To convert polar coordinates (r, θ) to rectangular coordinates (x, y) , let

$$\cos \theta = \frac{x}{r} \rightarrow x = r \cos \theta$$

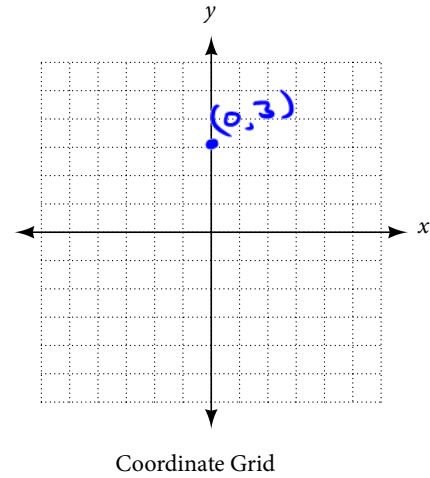
$$\sin \theta = \frac{y}{r} \rightarrow y = r \sin \theta$$

Example 2 Write the polar coordinates $(3, \frac{\pi}{2})$ as rectangular coordinates.

Solution

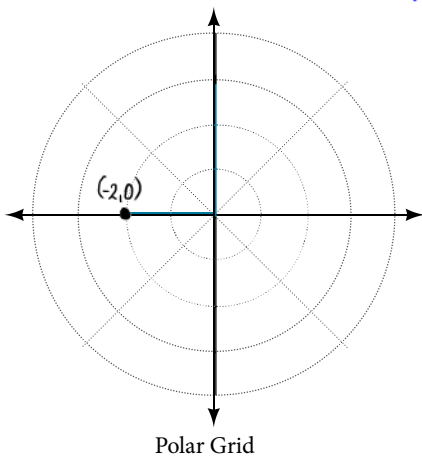


$$\begin{aligned}x &= r \cos \theta \\ &= 3 \cos\left(\frac{\pi}{2}\right) \\ &= 3 \cdot 0 \\ &= 0 \\ y &= r \sin \theta \\ &= 3 \sin\left(\frac{\pi}{2}\right) \\ &= 3 \cdot 1 \\ &= 3\end{aligned}$$

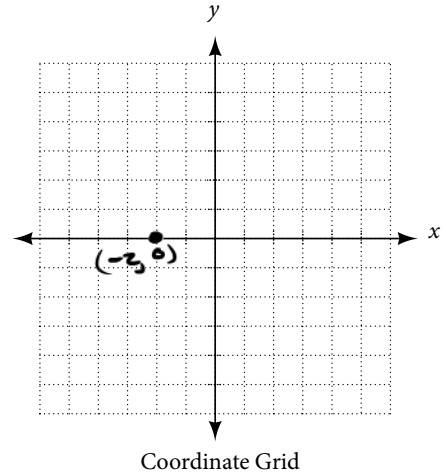


Example 3 Write the polar coordinates $(-2, 0)$ as rectangular coordinates.

Solution



$$\begin{aligned}x &= r \cos \theta \\ &= -2 \cdot \cos(0) \\ &= -2 \cdot (1) \\ &= -2 \\ y &= r \sin \theta \\ &= -2 \cdot \sin(0) \\ &= -2 \cdot 0 \\ &= 0\end{aligned}$$



Exercise: Write the polar coordinates $(-1, \frac{2\pi}{3})$ as rectangular coordinates.

$$\begin{aligned}x &= r \cos \theta \\ &= -1 \cdot \cos\left(\frac{2\pi}{3}\right) \\ &= -1 \cdot \left(-\frac{1}{2}\right) \\ &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}y &= r \sin \theta \\ &= -1 \cdot \sin\left(\frac{2\pi}{3}\right) \\ &= -1 \cdot \left(\frac{\sqrt{3}}{2}\right) \\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

$$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

Converting from Rectangular Coordinates to Polar Coordinates

To convert rectangular coordinates to polar coordinates, we will use two other familiar relationships.

With this conversion, we need to be aware that a set of rectangular coordinates will yield more than one polar point.

Converting from Rectangular Coordinates to Polar Coordinates

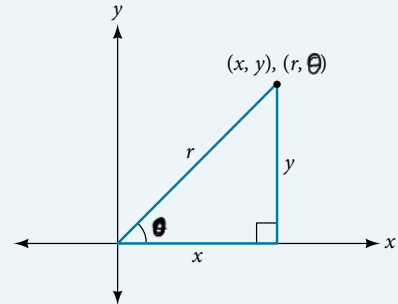
Converting from rectangular coordinates to polar coordinates requires the use of one or more of the relationships illustrated in **the Figure**.

$$\cos \theta = \frac{x}{r} \text{ or } x = r \cos \theta$$

$$r^2 = x^2 + y^2$$

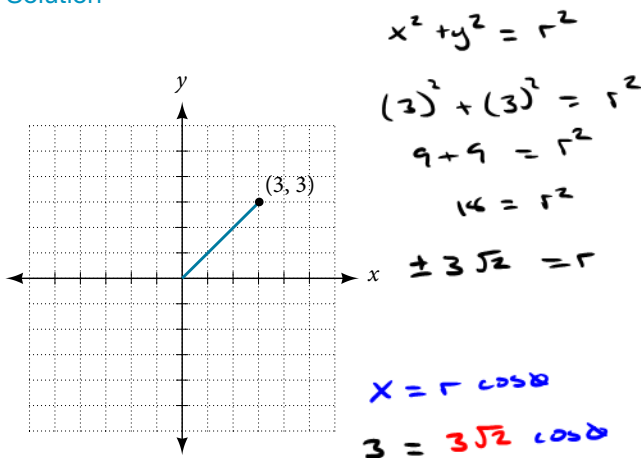
$$\sin \theta = \frac{y}{r} \text{ or } y = r \sin \theta$$

$$\tan \theta = \frac{y}{x}$$



Example 5 Convert the rectangular coordinates (3, 3) to polar coordinates.

Solution



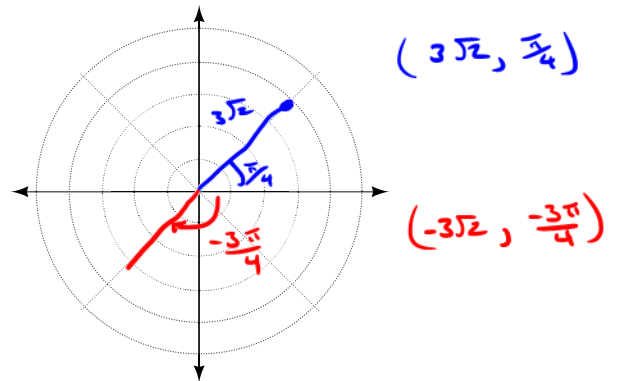
$$x = r \cos \theta$$

$$3 = 3\sqrt{2} \cos \theta$$

$$\frac{3}{3\sqrt{2}} = \cos \theta$$

$$\frac{1}{\sqrt{2}} = \cos \theta$$

$$\theta = \frac{\pi}{4}$$



26 - 30 E

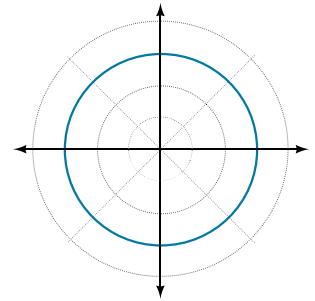
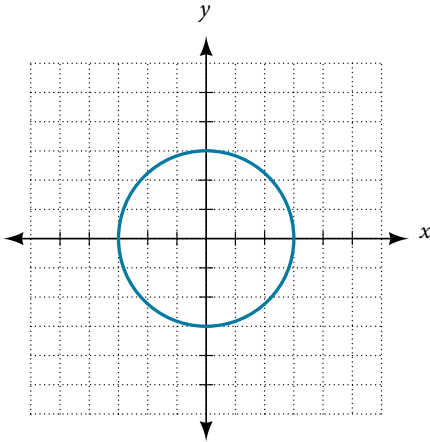
BYE
~

Transforming Equations between Polar and Rectangular Forms

We can now convert coordinates between polar and rectangular form. Converting equations can be more difficult, but it can be beneficial to be able to convert between the two forms. Since there are a number of polar equations that cannot be expressed clearly in Cartesian form, and vice versa, we can use the same procedures we used to convert points between the coordinate systems.

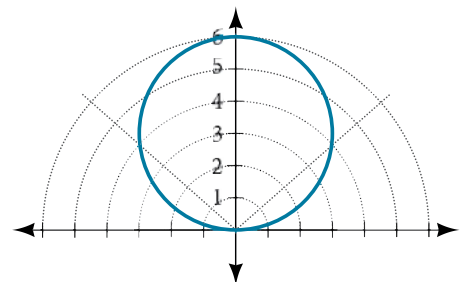
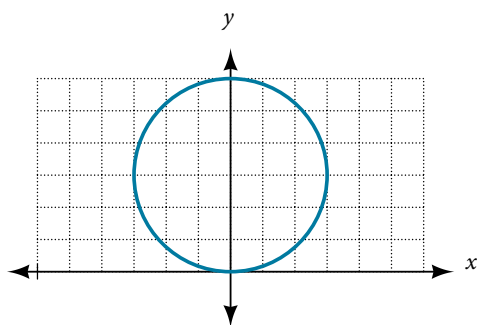
Example 6 Write the Cartesian equation $x^2 + y^2 = 9$ in polar form.

Solution



Example 7 Rewrite the Cartesian equation $x^2 + y^2 = 6y$ as a polar equation.

Solution



Example 8

- Rewrite the Cartesian equation $y = 3x + 2$ as a polar equation.
- Rewrite the Cartesian equation $y^2 = 3 - x^2$ in polar form.

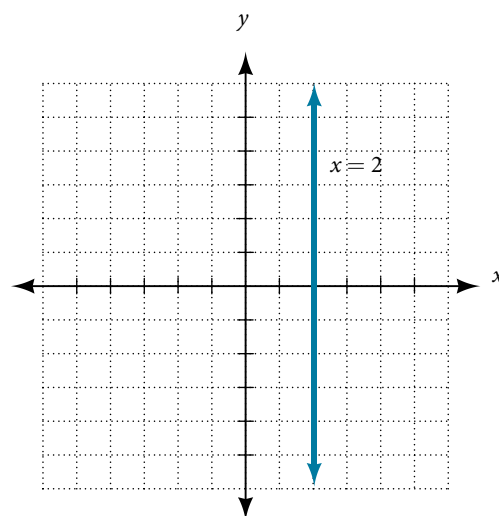
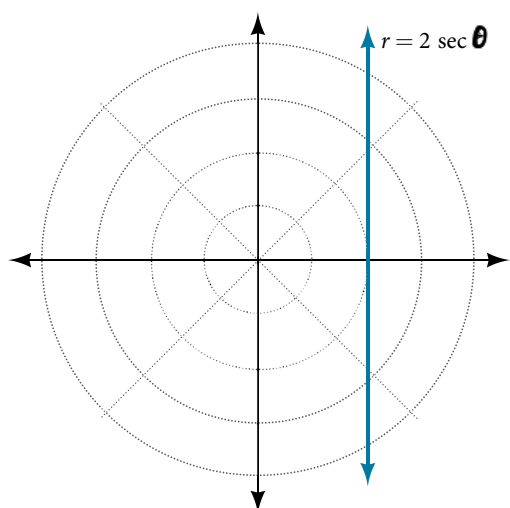
Solution

Identify and Graph Polar Equations by Converting to Rectangular Equations

We have learned how to convert rectangular coordinates to polar coordinates, and we have seen that the points are indeed the same. We have also transformed polar equations to rectangular equations and vice versa. Now we will demonstrate that their graphs, while drawn on different grids, are identical.

Example 9 Convert the polar equation $r = 2\sec \theta$ to a rectangular equation, and draw its corresponding graph.

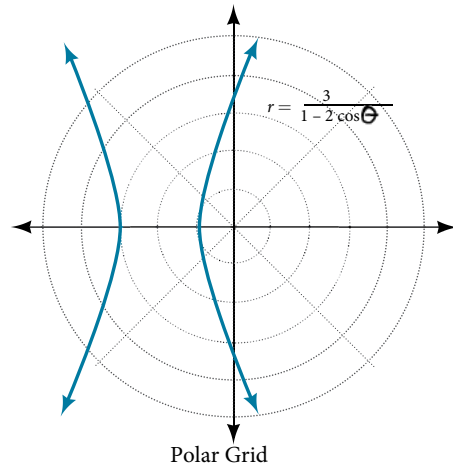
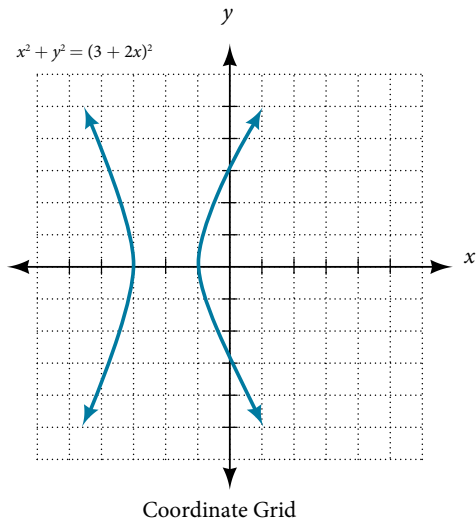
Solution



Exercise: Convert the polar equation $r = 2\csc \theta$ to a rectangular equation, and draw its corresponding graph.

Example 10 Rewrite the polar equation $r = \frac{3}{1 - 2\cos \theta}$ as a Cartesian equation.

Solution



Exercise: Rewrite the polar equation $r = 2\sin \theta$ in Cartesian form.

Example 11 Rewrite the polar equation $r = \sin(2\theta)$ in Cartesian form.

Solution

MORE EXERCISES

For the following exercises, convert the given Cartesian equation to a polar equation.

$$x = 3$$

$$x^2 + y^2 = 4y$$

$$x^2 + y^2 = 9$$

$$y = 4$$

$$x^2 + y^2 = 3x$$

$$x^2 = 9y$$

$$y = 4x^2$$

$$x^2 - y^2 = x$$

$$y^2 = 9x$$

$$y = 2x^4$$

$$x^2 - y^2 = 3y$$

$$9xy = 1$$

For the following exercises, convert the given polar equation to a Cartesian equation. Write in the standard form of a conic if possible, and identify the conic section represented.

$$r = 3\sin \theta$$

$$r = 2\sec \theta$$

$$r = 4$$

$$r = 4\cos \theta$$

$$r = 3\csc \theta$$

$$r = -10\sin \theta$$

$$r = \frac{4}{\sin \theta + 7\cos \theta}$$

$$r = \sqrt{r\cos \theta + 2}$$

$$r = \frac{1}{4\cos \theta - 3\sin \theta}$$

$$r = \frac{6}{\cos \theta + 3\sin \theta}$$

$$r^2 = 4\sec \theta \csc \theta$$

$$r = \frac{3}{\cos \theta - 5\sin \theta}$$

$$\theta = -\frac{2\pi}{3}$$

$$\theta = \frac{\pi}{4}$$