Homework 4

1. A spherical wave is a solution of the three-dimensional wave equation of the form $U(r,t)$, where r is the distance to the origin (the spherical coordinate) The wave equation takes the form

$$U_{tt} = c^2\left(U_{rr} + \frac{2}{r}U_r\right) \quad \text{(spherical wave equation)}$$

a). Solve the equation (Hint: Use the change of variable $V = rU$).

b). Solve the spherical wave equation with initial conditions $U(r,0) = \phi(r)$ and $U_t(r,0) = \psi(r)$ taking both $\phi(r)$ and $\psi(r)$ to be even functions of r.

2. Solve

$$\begin{cases} U_{xx} + U_{xt} - 2U_{tt} = 0 \\ U(x,0) = \phi(x),\ U_t(x,0) = \psi(x) \end{cases}$$

3. For the damped string,

$$u_{tt} - c^2 u_{xx} + r u_t = 0, \quad \text{for } r > 0$$

Show that the energy decreases.

4. Consider a solution of the diffusion equation $u_t = u_{xx}$ in $0 \le x \le L$, $0 \le t < +\infty$.

i). Let $M(T)$ be the maximum of $u(x,t)$ in the rectangle $\{0 \le x \le L, 0 \le t \le T\}$. Does $M(T)$ increase or decrease as a function of T?

ii) Let $m(T)$ be the minimum of $u(x,t)$ in the rectangle $\{0 \le x \le L, 0 \le t \le T\}$. Does $m(T)$ increase or decrease as a function of T?

5. Consider the diffusion equation

$$u_t = u_{xx} \qquad 0 < x < 1, \quad 0 < t < +\infty$$

(BC) $u(0,t) = u(1,t) = 0$ fn all t

(IC) $u(x,0) = 4x(1-x)$ fn $x \in [0,1]$

i) Show that $0 < u(x,t) < 1$ fn all $t > 0$
and $0 < x < 1$

ii) Show that.

$$u(x,t) = u(1-x,t) \quad \text{fn all } t \geq 0$$

and $0 < x < 1$

iii) Use the energy method to show that $\int_0^1 u^2 dx$ is strictly decreasing function of t.