Midterm I – MA 4335.

Due – 03/09/15 –

Consider the problem

$$\begin{cases} U_{tt} = c^2 U_{xx} & x \in \mathbb{R},\ t > 0 \\ U(x,0) = 0, \quad U_t(x,0) = \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| \geq a \end{cases} \end{cases}$$

for $c > 0$ and some $a > 0$.

i) Find analytically the solution $U(x,t)$.

ii) Let $a = 1$. For $c = 1$ and $c = 100$ plot the profile of U at succesive time instants
$t = \frac{a}{2c}, \frac{a}{c}, \frac{3a}{2c}, \frac{2a}{c}, \frac{5a}{c}$

Perform also a surface plot of $U(x,t)$ as a function of (x,t). Discuss the results and compare the two situations $c = 1$ and $c = 10$.

iii) Find $\max\limits_{x \in \mathbb{R}} U(x,t)$ as a function of t.

and prove that the maximum principle for the wave equation does not hold!

2. Prove the uniqueness of the following problem by using the energy method.

$$\begin{cases} U_t = kU_{xx} & 0 < x < L,\ t > 0 \\ U(x,0) = \phi(x),\ U_x(0,t) = g(t),\ U_x(L,t) = h(t) \end{cases}$$

where g and h are smooth function of time and ϕ is a smooth function of space.

3. Consider the following problem

$$\begin{cases} U_t - kU_{xx} + bU = 0 & x \in \mathbb{R},\ t > 0 \\ U(x,0) = \phi(x) \end{cases}$$

and $b > 0$ constant

i) Solve the problem to find $U(x,t)$
(Hint: Use change of variable $V(x,t) = e^{bt} U(x,t)$).

ii) For $b = 1$ and $b = 10$ and for $k = 1$ with

$$\phi(x) = \begin{cases} 1 & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

plot the solution on a surface plot.
Compare and discuss the results.

iii) Discuss numerically what happens when k gets larger or very small. For this part fix $b=1$ and consider

$$\phi(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| \geq 1. \end{cases}$$

4. Solve

$$\begin{cases} U_{xx} - 3U_{xt} - 4U_{tt} = 0 \\ U(x,0) = x^2, \quad U_t(x,0) = e^x \end{cases}$$

and plot the solution $U(x,t)$ numericaly on a surface plot.

(Hint: Factor the operator as we did for the wave equation)