Midterm 2

MA 4335

Due 04/24/15

Total points 125

Maximum needed 100

1. Solve

20p
$$\begin{cases} U_{tt} = 4U_{xx} & \text{in } 0 < x < \infty \\ U(x,0) = 0; \quad U_t(x,0) = 1; \\ U_t(0,t) + 3U_x(0,t) = 0. \end{cases}$$

Discuss the discontinuity at origin.! How does it propagate. ?

5p Plot the solution for several time values (e.g. $t=1$, $t=2$, $t=3$, $t=4$).

2. Solve

20p
$$U_{tt} = 9U_{xx} \qquad 0 < X < \infty$$
$$U(0,t) = t^2, \quad U(x,0) = x, \quad U_t(x,0) = 0$$

5p Plot your solution.

3. Solve.

20 pts

$$\begin{cases} u_{tt} = u_{xx} & \text{in } (0,1) \\ u(0,t) = 6 \ ; \ u(1,t) = 3 \\ u(x,0) = \sin \pi x + 3\sin 7\pi x + 6 - 3x \\ u_t(x,0) = 0 \end{cases}$$

5 pts Plot your solution.

4. Solve.

$$\begin{cases} u_t = 2u_{xx} + x^2 - x - 4t & \text{in } (0,1) \\ u(0,t) = 0 = u(1,t) \\ u(x,0) = \sin \pi x + 6\sin 8\pi x \end{cases}$$

24 pts

5 pts Plot your solution.

Problem 1

General form of solution

5p. $U(x,t) = f(x+2t) + g(x-2t)$, $x>0, t\geq 0$

Thus
$U_t(x,t) = 2f'(x+2t) - 2g'(x-2t)$, $x>0, t\geq 0$

$t=0$ implies

$$(1)\begin{cases} f(x)+g(x)=0 \\ 1 = 2f'(x) - 2g'(x) \end{cases} \quad \text{for } x>0.$$

Differentiate the first equation in (1) (multiply by 2) and add to the second to obtain

$$\begin{cases} f'(x) = \frac{1}{4} \\ \text{for } x>0 \end{cases} \Longrightarrow \begin{cases} g'(x) = -\frac{1}{4} \end{cases} \quad \text{for } x>0.$$

Thus

$$(2)\begin{cases} f(x) = \frac{1}{4}x + A \quad , \text{ for } x>0 \text{ and } A\in\mathbb{R} \\ g(x) = -\frac{1}{4}x + B \quad , \text{ for } x>0 \text{ and } B\in\mathbb{R} \end{cases}$$

4p.

with $A+B=0$ (3) 1p

The boundary condition yields.

$$0 = U_t(0,t) + 3U_x(0,t) = \left(2f'(2t) - 2g'(-2t)\right) + $$
$$+ 3\left(f'(2t) + g'(-2t)\right) \quad 2p$$

$$= 5f'(2t) + g'(-2t) \qquad \text{for } t>0$$

Thus

$$g'(-2t) = -5f'(2t) \qquad \text{for } t>0$$

This means that.

$$g'(-y) = -5f'(y) \qquad \text{for } y>0$$

$$\Updownarrow$$

$$g(-y) = -5f(y) + k \qquad \text{for } y>0 \text{ and } k \in \mathbb{R}$$

$$\Updownarrow \text{ from } (2)$$

3p (4) $g(-x) = -5\left(\frac{1}{4}x + A\right) + k$, for $x>0$ and some $k \in \mathbb{R}$.

g is continuous at $x=0$ and this implies from (2), $-5A + k = B$ (5) 2p

From (3) and (5) we see

$$k = 4A \qquad (6)$$

Thus

$$U(x,t) = f(x+2t) + g(x-2t)$$

$$= \begin{cases} t & \text{for } x > 2t \quad (\text{From } (2)) \\ (3t-x) & \text{for } x < 2t \quad (\text{From } (2)+(4)+(5)) \end{cases}$$

3p

The point $(0,0)$ is a point of discontinuity

1p. Since $(U_t + 3U_x)(0,0) = 1 \neq 0$ (Contradiction with the boundary cond. at $x=0$)

The discontinuity will propagate along the characteristic line $x = 2t$.

Plot - Optional — 5p.

Problem 2

Homogenise the data. Consider

$$V(x,t) = U(x,t) - t^2 \qquad 3p$$

V satisfies.

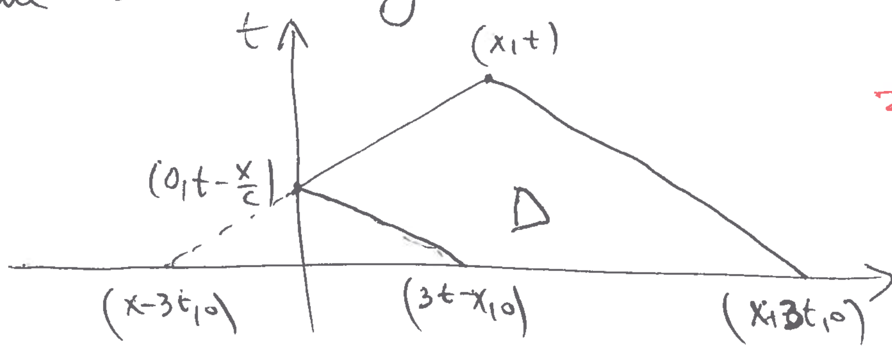
$$2p \quad \begin{cases} V_{tt} - 9V_{xx} = -2 & 0<x<\infty,\ 0<t<\infty \\ V(x,0) = x & ,0<x<\infty \\ V_t(x,0) = 0 & ,0<x<\infty \\ V(0,t) = 0 & ,0<t<\infty. \end{cases}$$

Formula (19), Section 3.4 gives the solution.

For $x>3t$

$$V(x,t) = x + \frac{1}{6}\int_0^t \int_{x-3(t-s)}^{x+3(t-s)} -2\, dy\, ds = x - t^2 \qquad 5p$$

For $x<3t$ Domain of dependence as in Figure 3.2.2. in your books.



2p

$$V(x,t) = \frac{1}{2}\left(3t+x-(3t-x)\right) + \frac{1}{6}\iint_{6} -2\, dA =$$

$$= x - \frac{1}{3}\left(2tx - \frac{x^2}{3}\right) = x - \frac{2tx}{3} + \frac{x^2}{9}.$$ 3p

So

$$u(t,x) = \begin{cases} x & x > 3t \\ x - \frac{2tx}{3} + \frac{x^2}{9} + t^2, & x < 3t \end{cases}$$ 5p

When we recall that $u(x,t) = V(x,t) + t^2$.

Plot optional — 5p

Problem 3

Homogenise the data!

Consider

$$V(x,t) = U(x,t) - (6-3x) \qquad (1) \quad 3p$$

V satisfies

$$2p \quad \begin{cases} V_{tt} = V_{xx} & \text{in } (0,1) \\ V(0,t) = 0 = V(1,t) \\ V(x,0) = \sin \pi x + 3 \sin 7\pi x \\ V_t(x,0) = 0 . \end{cases}$$

Separation of variables gives.

$$V(x,t) = \sum_{n=1}^{\infty} A_n \cos n\pi t \ \sin n\pi x \qquad 2p$$

From initial data we obtain

$$V(x,0) = \sin \pi x + 3 \sin 7\pi x = \sum_{n=1}^{\infty} A_n \sin n\pi x \qquad 3p \quad (2)$$

5p $A_1 = 1$, $A_7 = 3$ and $A_n = 0$ for $n \neq 1, n \neq 7$.

Thus from (1) and (2) we have that the solution to the problem is

5p $$u(x,t) = \left(\cos \pi t \sin \pi x + 3 \cos 7\pi t \sin 7\pi x \right) + 6 - 3x$$

Plots Optional — 5p

Problem 4

Homogenise the data! Consider

$$V(x,t) = U(x,t) - x(x-1)t. \quad (1) \quad 3p$$

V satisfies. (check!)

$$2p \begin{cases} V_t - 2V_{xx} = 0 & \text{in } (0,1) \\ V(0,t) = 0 = V(1,t) & \text{for } t>0 \\ V(x,0) = \sin \pi x + 6 \sin 8\pi x & \text{in } (0,1). \end{cases}$$

Separation of variable gives.

$$V(x,t) = \sum_{n=1}^{\infty} A_n e^{-2(n\pi)^2 \cdot t} \sin n\pi x \quad (2) \quad 2p$$

From $V(x,0) = \sin \pi x + 6 \sin 8\pi x$ we obtain 3p

5p. $A_1 = 1$ and $A_8 = 6$ with $A_n = 0$ for $n \neq 1, n \neq 8$

Thus from (1) and (2) we get that the solution U to our problem is given by

5p $U(x,t) = e^{-2\pi^2 t} \sin \pi x + 6e^{-128\pi^2 t} \sin 8\pi x + x(x-1)t$

Plots Optional - 5p

Problem 5

Separation of variable gives

$$U(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(n)^2 \cdot t} \cos nx \qquad 6p.$$

$$A_n = \frac{2}{\pi}\int_0^{\pi}\left(-\frac{x^3}{3}+\frac{x^2}{2}\pi\right)\cos nx \, . =$$

$$= \frac{2}{\pi}\left[\left(-\frac{x^3}{3}+\frac{x^2}{2}\pi\right)\frac{\sin nx}{n}\right]\Bigg|_0^{\pi} -$$

$$-\frac{2}{\pi}\int_0^{\pi}\left(-x^2+x\pi\right)\frac{\sin nx}{n}dx =$$

$$= -\frac{2}{n\pi}\left[\left(-x^2+x\pi\right)\left(-\frac{\cos nx}{n}\right)\right]\Bigg|_0^{\pi}$$

$$-\frac{2}{n^2\pi}\int_0^{\pi}\left(-2x+\pi\right)\cos nx \, dx$$

$$= -\frac{2}{n^2\pi}\left(\left(-2x+\pi\right)\frac{\sin nx}{n}\right)\Bigg|_0^{\pi} + \frac{4}{n^3\pi}\int_0^{\pi}\sin nx \, dx$$

Thus

$$A_n = \frac{4}{n^3 \pi} \cos nx \Big|_0^{\pi} = \frac{4}{n^4 \pi}\left((-1)^n - 1\right), \quad n \neq 0.$$

6p

$$A_0 = \frac{2}{\pi} \int_0^{\pi} \left(-\frac{x^3}{3} + \frac{x^2}{2}\pi\right) =$$

$$= \frac{2}{\pi}\left(-\frac{x^4}{12} + \frac{x^3}{6}\pi\right)\Big|_0^{\pi} = \frac{\pi^3}{6}.$$

3p

Thus

$$U(x,t) = \frac{\pi^3}{12} + \sum_{n=1}^{\infty} \frac{4}{n^4 \pi}\left((-1)^n - 1\right) e^{-n^2 t} \cos nx$$

5p

$$= \frac{\pi^3}{12} + \sum_{n \text{ odd}} \frac{(-8)}{n^4 \pi} e^{-n^2 t} \cos nx.$$

Plots Optional - 5p