Example 4 $y = e^{ix}$ satisfies $y'' + y = 0$, for

$$y' = ie^{ix} \qquad y'' = -e^{ix} \qquad e^{ix} - e^{ix} \equiv 0$$

The following theorem shows the connection between real and complex solutions of a linear differential equation with *real* coefficients.

Theorem 1 *Consider the differential equation*

$$a(x)y'' + b(x)y' + c(x)y = 0$$

where a, b, c are real functions. The complex function $y = u + iv$ where u, v are real is a solution if and only if u and v are solutions.

Proof As usual, denote the left-hand side of the differential equation by $L(y)$. It is easy to prove (see Prob. 3) that $L(y) = L(u) + iL(v)$ where $L(u)$ and $L(v)$ are real. Therefore y is a solution if and only if $L(y) = L(u) + iL(v) \equiv 0$. Since a complex number is zero if and only if its real and imaginary parts are zero, we have $L(y) = 0$ if and only if $L(u) = 0$ and $L(v) = 0$.

Example 5 In Example 4 we have seen that $y = e^{ix}$ satisfies $y'' + y = 0$. Since $e^{ix} = \cos x + i \sin x$, the above theorem shows that $\cos x$ and $\sin x$ are real solutions.

Problems 5-3

1. If $f = u + iv$ and $g = r + is$ are differentiable complex functions of a real variable, prove that:

a. $(f + g)' = f' + g'$

b. $(fg)' = fg' + f'g$

c. $\left(\dfrac{f}{g}\right)' = \dfrac{gf' - fg'}{g^2}$

d. $(cf)' = cf'$ c a complex constant

2. Verify that e^{cx} where c is complex and x is real satisfies:

a. $e^{cx_1}e^{cx_2} = e^{c(x_1+x_2)}$

b. $\dfrac{e^{cx_1}}{e^{cx_2}} = e^{c(x_1-x_2)}$

c. Recalling De Moivre's formula (n a positive integer)

$(\cos bx + i \sin bx)^n = \cos nbx + i \sin nbx$, show that $(e^{cx})^n = e^{cxn}$.

3. If $y = u + iv$ and $Ly = a(x)y'' + b(x)y' + c(x)y$, where a, b, c are real functions, show that $L(y) = L(u) + iL(v)$.

4. Show that $y = e^{(1+i)x}$ satisfies $y'' - 2y' + 2y = 0$. What are real solutions of the equation?

5. Define $x^{(a+ib)} = e^{(a+ib)\ln x}$ for real $x > 0$.

a. Show that $x^{a+ib} = x^a[\cos(b \ln x) + i \sin(b \ln x)]$.

b. Show that

$$\frac{d}{dx}x^{a+ib} = (a + ib)\,x^{(a-1)+ib}$$

(i.e., the usual rule for differentiation holds).

6. Show that $y = x^{1+i}$ satisfies $x^2y'' - xy' + 2y = 0$. What are real solutions of the equation?

7. Show that $y = e^{ix}$ satisfies $y' - iy = 0$. Are the real and imaginary parts of y solutions of the equation? Why?

8. Let u and v be real linearly independent solutions of $a(x)y'' + b(x)y' + c(x)y = 0$ where a, b, c are real functions. Show that $y_1 = u + iv$ and $\bar{y}_1 = u - iv$ ($\bar{y}$ is the complex conjugate of y) are complex solutions. Show that the general complex solution is $y = c_1y_1 + c_2\bar{y}_1$ where c_1 and c_2 are arbitrary complex numbers.

5-4 Homogeneous linear equations with constant coefficients

A simple, but important, class of linear differential equations is that with constant coefficients. We consider the homogeneous equation

$$ay'' + by' + cy = 0 \tag{8}$$

where a, b, and c are *real* constants and $a \neq 0$. Since this equation does not contain x explicitly, it could be solved by the method of Sec. 2-7. We prefer to use a simpler, but less direct, approach.

Let us consider the types of functions that could possibly satisfy Eq. (8). The solution could not be a function like $y = \ln x$, for the derivatives of this function are $y' = 1/x$ and $y'' = -1/x^2$. Upon substitution into the equation there would be no term to cancel the term $c(\ln x)$. It is clear that the solution must be a function whose derivative does not differ greatly in form from the function itself. The functions $e^{\lambda x}$, $\sin \beta x$, $\cos \beta x$ come to mind immediately. The functions $\sin \beta x$ and $\cos \beta x$ are simply combinations of complex exponentials and need not be considered separately. Therefore it is reasonable to look for solutions in the form

$$y = e^{\lambda x} \tag{9}$$

where λ must be determined so that (9) satisfies the differential equation (8). The derivatives of (9) are $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$. Substituting these into Eq. (8), we find

$$e^{\lambda x}(a\lambda^2 + b\lambda + c) \equiv 0$$

If this equation holds for all x, then $e^{\lambda x}$ is a solution for all x; however, the above can hold for all x only if

$$a\lambda^2 + b\lambda + c = 0 \tag{10}$$

This equation, which determines λ, is called the *auxiliary* or *characteristic* equation. The roots of (10) are

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the discriminant $\Delta = b^2 - 4ac$ is greater than zero, the roots are real and distinct; if Δ is equal to zero, the roots are real and equal; and if Δ is less than zero, the roots are complex conjugate numbers. We discuss each of these cases.

Case I. *Real, distinct roots,* $\Delta > 0$. If $\Delta > 0$, the roots are

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

and are real, distinct numbers. Therefore $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are both solutions of the differential equation. Since these functions are obviously linearly independent, the general solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \tag{11}$$

Case II. *Real, equal roots,* $\Delta = 0$. In this case we have only one root given by

$$\lambda_1 = -\frac{b}{2a} \tag{12}$$

Therefore $e^{\lambda_1 x}$ is a solution of the differential equation. Of course, $c_1 e^{\lambda_1 x}$ is also a solution, but we need a second linearly independent solution. To find this, we use the method of variation of parameters.† Assume a solution of the form $y = v(x)e^{\lambda_1 x}$. Substituting this into the differential equation (8), we obtain

$$av'' + (2a\lambda_1 + b)v' + (a\lambda_1^2 + b\lambda_1 + c)v = 0 \tag{13}$$

† This method, when used in this manner to find a second solution of a homogeneous linear differential equation, when one solution is known, is often called *reduction of order* (see Sec. 5-6, Prob. 9).

Since λ_1 is a root of the characteristic equation, the coefficient of v vanishes, and since $\lambda_1 = -b/2a$, the coefficient of v' also vanishes. The differential equation (13) therefore reduces to

$$v'' = 0 \tag{14}$$

The general solution of this equation is

$$v = c_1 + c_2 x$$

Since $y = v(x)e^{\lambda_1 x}$, we have that

$$y = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x} \tag{15}$$

is a solution of the differential equation (8). Since the functions $e^{\lambda_1 x}$ and $xe^{\lambda_1 x}$ are linearly independent, the general solution is given by Eq. (15).

Case III. *Complex conjugate roots,* $\Delta < 0$. In this case the roots are

$$\lambda_1 = \alpha + i\beta = -\frac{b}{2a} + i\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}$$

$$\lambda_2 = \alpha - i\beta = -\frac{b}{2a} - i\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}$$

and are conjugate complex numbers (since a, b, c are real). These yield the two complex solutions†

$$e^{(\alpha+i\beta)x} = e^{\alpha x}(\cos \beta x + i \sin \beta x) \tag{16}$$

$$e^{(\alpha-i\beta)x} = e^{\alpha x}(\cos \beta x - i \sin \beta x) \tag{17}$$

Since the differential equation (8) has *real* coefficients, the real and imaginary parts of either of these functions, namely,

$$e^{\alpha x} \cos \beta x \qquad e^{\alpha x} \sin \beta x \tag{18}$$

are real solutions (see Theorem 1, Sec. 5-3). The Wronskian of these functions is easily shown to be nonzero. Therefore the functions (18) are linearly independent solutions and the general solution for this case is

$$y = e^{\alpha x}(A \cos \beta x + B \sin \beta x) \tag{19}$$

where A, B are arbitrary real constants.

We also note that the general complex-valued solution of the differential equation (8) is (see Prob. 8, Sec. 5-3)

$$y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} \tag{20}$$

† See Sec. 5-3.

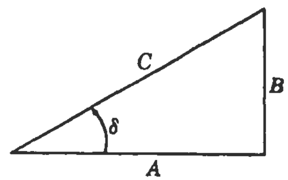

figure 1

where c_1 and c_2 are *arbitrary complex* constants. Equation (20) can be written, using the right-hand sides of (16) and (17),

$$y = e^{\alpha x}[(c_1 + c_2)\cos\beta x + i(c_1 - c_2)\sin\beta x] \tag{21}$$

If we now take c_1 and c_2 to be *conjugate* complex constants, then it is easily shown that $c_1 + c_2$ and $i(c_1 - c_2)$ are both real numbers. Calling these numbers A and B respectively, we have again the real general solution (19).

It is often convenient to write the solution for this case in still another form. Let A and B be thought of as the coordinates of a point in the plane. Introducing polar coordinates C and δ of the point, we have (see Fig. 1)

$$\begin{aligned} A &= C\cos\delta \\ B &= C\sin\delta \end{aligned} \qquad \text{or} \qquad \begin{aligned} C &= \sqrt{A^2 + B^2} \\ \delta &= \arctan\frac{B}{A}\,\dagger \end{aligned} \tag{22}$$

Substituting these in Eq. (19), we obtain

$$y = Ce^{\alpha x}(\cos\delta\cos\beta x + \sin\delta\sin\beta x) \tag{23}$$

Recalling the addition law for cosines, we obtain the solution in the compact form

$$y = Ce^{\alpha x}\cos(\beta x - \delta) \tag{24}$$

In a similar way we obtain

$$y = Ce^{\alpha x}\sin(\beta x + \theta) \tag{25}$$

where the angle θ is the complement of δ.

Summary *The solutions of*

DE: $ay'' + by' + cy = 0 \qquad a \neq 0$

† By the angle $\delta = \arctan B/A$ we do not mean the principal value of arctan B/A but rather the angle δ such that

$$\sin\delta = \frac{B}{\sqrt{A^2 + B^2}} \qquad \text{and} \qquad \cos\delta = \frac{A}{\sqrt{A^2 + B^2}}$$

are obtained by first solving the characteristic equation

$$a\lambda^2 + b\lambda + c = 0$$

and, according to the type of roots, using the results of the accompanying table.

Type of root	Linearly independent solutions	General solution
Real, distinct $\lambda_1 \neq \lambda_2$	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$
Real, equal $\lambda_1 = \lambda_2$	$e^{\lambda_1 x}, xe^{\lambda_1 x}$	$c_1e^{\lambda_1 x} + c_2xe^{\lambda_1 x}$
Complex conjugate $\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$ $\beta \neq 0$	$e^{(\alpha+i\beta)x}, e^{(\alpha-i\beta)x}$ or $e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x$	$c_1e^{(\alpha+i\beta)x} + c_2e^{(\alpha-i\beta)x}$ or $e^{\alpha x}(A\cos\beta x + B\sin\beta x)$ or $Ce^{\alpha x}\cos(\beta x - \delta)$ or $Ce^{\alpha x}\sin(\beta x + \theta)$

Most scientists and engineers encounter linear equations with constant coefficients so often that the above results, and the methods used to obtain them, are committed to memory.

Example 1 $\quad 2y'' - y' - 3y = 0$

The characteristic equation is

$$2\lambda^2 - \lambda - 3 = 0$$

Therefore

$$\lambda_1 = -1 \qquad \lambda_2 = \tfrac{3}{2}$$

and the general solution is

$$y = c_1e^{-x} + c_2e^{3x/2}$$

Example 2 $\quad y'' - 4y' + 4y = 0$

The characteristic equation is

$$\lambda^2 - 4\lambda + 4 = 0$$

$$\lambda = 2, 2$$

The general solution is

$$y = (c_1 + c_2x)e^{2x}$$

Example 3 $32y'' - 40y' + 17y = 0$

The characteristic equation is

$$32\lambda^2 - 40\lambda + 17 = 0$$

$$\lambda_1 = \tfrac{5}{8} + i\tfrac{3}{8} \qquad \lambda_2 = \tfrac{5}{8} - i\tfrac{3}{8}$$

The solution can be written in the form

$$y = c_1 e^{(\frac{5}{8}+i\frac{3}{8})x} + c_2 e^{(\frac{5}{8}-i\frac{3}{8})x}$$

or

$$y = e^{\frac{5}{8}x}(A \cos \tfrac{3}{8}x + B \sin \tfrac{3}{8}x)$$

or

$$y = ce^{\frac{5}{8}x} \cos (\tfrac{3}{8}x - \delta)$$

Problems 5-4

1. Find the general solution of:

a. $y'' = 0$ b. $y'' - 2y' = 0$
c. $y'' - a^2y = 0$ d. $y'' + a^2y = 0$
e. $y'' + y' = 0$ f. $y'' + 2y' + y = 0$
g. $3y'' + 14y' + 8y = 0$ h. $y'' + y' + y = 0$

2. Suppose the characteristic equation for differential equation (8) has distinct, real roots, λ_1 and λ_2. Show that $(e^{\lambda_1 x} - e^{\lambda_2 x})/(\lambda_1 - \lambda_2)$ is a solution of Eq. (8). Show also that

$$\lim_{\lambda_2 \to \lambda_1} \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{\lambda_1 - \lambda_2} = xe^{\lambda_1 x}$$

This limiting process is another way of finding the second linearly independent solution in the case of equal roots.

3. Suppose the roots of the characteristic equation are real and distinct. The roots can then be written as $\lambda_{1,2} = A \pm B$ where A and B are real. Show that the solution of the differential equation can be written as $y = e^{Ax}(c_1 \cosh Bx + c_2 \sinh Bx)$. Recall the definitions of the hyperbolic functions:

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

4. Write down a homogeneous second-order linear differential equation with constant real coefficients whose solutions are:

a. $1, x$ b. $2e^x, e^{-5x}$
c. $3 \cos 4x, 5 \sin 4x$ d. $e^{-x} \cos x, e^{-x} \sin x$
e. $5 \cosh 2x, 9 \sinh 2x$ f. $4e^{3x}, 7xe^{3x}$
g. $e^x \cos (x - 1), e^x \cos (x - 2)$ h. x, e^x

5. We note that the solutions of the second-order differential equation with constant coefficients are defined for all x. Could this have been predicted from the existence theorem?

6. Referring to the existence theorem, explain why the function $\ln x$ cannot be the solution of a homogeneous linear equation with constant coefficients.

7. Solve: DE: $y'' + y' - 6y = 0$
IC: $y(0) = 1 \qquad y'(0) = -2$

8. Solve the following third-order equations by extending the method in the text:

a. $y''' = 0$ b. $y''' - y = 0$
c. $y''' - 7y'' + 16y' - 12y = 0$ d. $y''' - 3y'' + 3y' - y = 0$

5-5 *Undetermined coefficients*

We have seen that the general solution of the nonhomogeneous equation

$$L(y) = ay'' + by' + cy = f(x) \qquad a \neq 0 \tag{26}$$

is

$$y = y_h + y_p \tag{27}$$

where y_h is the general solution of the homogeneous equation and y_p is a particular solution of Eq. (26). In Sec. 5-4 we solved the homogeneous equation with constant coefficients. We now investigate methods of finding particular solutions of Eq. (26) when $f(x)$ is an exponential, a sinusoid, a polynomial, or a product of such functions. These functions appear often in applications.

I. $f(x) = ke^{\alpha x}$. We seek a particular solution of the differential equation

$$ay'' + by' + cy = ke^{\alpha x} \tag{28}$$

Because of the exponential term on the right-hand side of the equation, we look for a solution in the form

$$y_p = Ae^{\alpha x} \tag{29}$$

where the undetermined coefficient A will be determined so that the differential equation is satisfied. The substitution of (29) into the differential equation (28) results in

$$(a\alpha^2 + b\alpha + c)Ae^{\alpha x} = ke^{\alpha x} \tag{30}$$

or

$$A = \frac{k}{a\alpha^2 + b\alpha + c}$$

Therefore

$$y_p = \frac{ke^{\alpha x}}{a\alpha^2 + b\alpha + c} \tag{31}$$

provided the denominator is not zero. We note that the denominator is the characteristic polynomial

$$p(\lambda) = a\lambda^2 + b\lambda + c \tag{32}$$

evaluated at $\lambda = \alpha$. The particular solution can therefore be written in the convenient form

$$y_p = \frac{ke^{\alpha x}}{p(\alpha)} \qquad p(\alpha) \neq 0 \tag{33}$$

If $p(\alpha) = 0$, Eq. (33) does not determine a particular solution. In this case α is a root of the characteristic equation. Therefore $e^{\alpha x}$ is a solution of the homogeneous equation and cannot possibly also be a solution of the nonhomogeneous equation. To find a particular solution if $p(\alpha) = 0$, we assume

$$y_p = Axe^{\alpha x}\dagger \tag{34}$$

Substituting into the differential equation (28), we obtain

$$(a\alpha^2 + b\alpha + c)Axe^{\alpha x} + (2a\alpha + b)Ae^{\alpha x} = ke^{\alpha x}$$

Since $p(\alpha) = a\alpha^2 + b\alpha + c = 0$, we have $A = k/(2a\alpha + b)$ and

$$y_p = \frac{kxe^{\alpha x}}{2a\alpha + b} \tag{35}$$

provided the denominator is not zero. We note that $2a\alpha + b = p'(\alpha)$; therefore

$$y_p = \frac{kxe^{\alpha x}}{p'(\alpha)} \qquad p(\alpha) = 0, p'(\alpha) \neq 0 \; . \tag{36}$$

If $p(\alpha) = 0$ and $p'(\alpha) \neq 0$, then α is a simple root of the characteristic equation. If both $p(\alpha)$ and $p'(\alpha)$ are zero, α is a double root of the characteristic equation.‡ This means that both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of the homogeneous equation. In this case we assume

$$y_p = Ax^2e^{\alpha x} \tag{37}$$

† Alternatively we could assume $y_p = v(x)e^{\alpha x}$ and determine $v(x)$ by substituting into (28).

‡ If α is a root of $p(\lambda) = 0$, then $p(\lambda) = (\lambda - \alpha)g(\lambda)$ where $g(\lambda)$ is a linear polynomial. If $g(\alpha) \neq 0$, α is a simple root and if $g(\alpha) = 0$, α is a double root. We see that $p'(\lambda) = (\lambda - \alpha)g'(\lambda) + g(\lambda)$; therefore $p'(\alpha) = g(\alpha)$. This shows that if α is a simple root, $p'(\alpha) \neq 0$ and if α is a double root, $p(\alpha) = p'(\alpha) = 0$.

Proceeding as above, we obtain

$$y_p = \frac{kx^2e^{\alpha x}}{2a} = \frac{kx^2e^{\alpha x}}{p''(\alpha)} \qquad p(\alpha) = p'(\alpha) = 0 \tag{38}$$

The denominator cannot be zero since, by assumption, a is different from zero. Summarizing the above we have the following convenient rule: *A particular solution of $L(y) = ke^{\alpha x}$ is given by*

$$y_p = \begin{cases} \dfrac{ke^{\alpha x}}{p(\alpha)} & p(\alpha) \neq 0 \\[2ex] \dfrac{kxe^{\alpha x}}{p'(\alpha)} & p(\alpha) = 0, \quad p'(\alpha) \neq 0 \\[2ex] \dfrac{kx^2e^{\alpha x}}{p''(\alpha)} & p(\alpha) = p'(\alpha) = 0 \end{cases}$$

Example 1 Solve

$$y'' - 5y' + 4y = 3 + 2e^x$$

$$p(\lambda) = \lambda^2 - 5\lambda + 4 = 0 \qquad \lambda = 4, 1$$

$$y_h = c_1e^x + c_2e^{4x}$$

$$y_p = \frac{3}{p(0)} + \frac{2xe^x}{p'(1)} = \tfrac{3}{4} - \tfrac{2}{3}xe^x$$

$$y = c_1e^x + c_2e^{4x} + \tfrac{3}{4} - \tfrac{2}{3}xe^x$$

II. $f(x) = k\cos\beta x$ *or* $f(x) = k\sin\beta x$. We shall handle this case by exploiting the relationship between sinusoids and complex exponentials. At first sight the method may appear artificial. However, the method is much used and is the basis of the impedance method used in solving alternating-current circuits.† Consider the differential equation

$$L(y) = ay'' + by' + cy = k\cos\beta x \tag{39}$$

Also consider the companion equation

$$L(v) = av'' + bv' + cv = k\sin\beta x \tag{40}$$

We now combine these two real equations into one equation for the complex function

$$w = y + iv \tag{41}$$

† See Secs. 3-9 and 6-4.

Since the operator L is linear, we have

$$L(w) = L(y) + iL(v)$$
$$= k(\cos \beta x + i \sin \beta x)$$

Therefore the complex function w satisfies

$$L(w) = ke^{i\beta x} \tag{42}$$

This equation has an exponential on the right-hand side and can be solved as in Case I. After w is found, y can be found by taking the real part of w, and v by taking the imaginary part. We summarize this method below.

Method *To find a particular solution of* $L(y) = k \cos \beta x$ [*or* $L(y) = k \sin \beta x$], *find a particular solution of* $L(w) = ke^{i\beta x}$ *and take the real part* [*or imaginary part*] *of the result.*

Example 2 Find a particular solution of

$$y'' + 7y' + 12y = 3 \cos 2x \tag{43}$$

We have

$$p(\lambda) = \lambda^2 + 7\lambda + 12 \tag{44}$$

Consider $y = \text{Re}\, w =$ real part of w, where w satisfies the equation

$$w'' + 7w' + 12w = 3e^{i2x} \tag{45}$$

A particular solution of Eq. (45) is

$$w_p = \frac{3e^{i2x}}{p(2i)} = \frac{3e^{i2x}}{8 + 14i} \tag{46}$$

In order to find $\text{Re}\, w_p = y_p$, we must put w_p in an appropriate form:

$$w_p = \frac{3e^{i2x}}{8 + 14i} \frac{8 - 14i}{8 - 14i}$$
$$= \tfrac{3}{260}(\cos 2x + i \sin 2x)(8 - 14i)$$
$$= \tfrac{3}{260}(8 \cos 2x + 14 \sin 2x) + \tfrac{3}{260}i(8 \sin 2x - 14 \cos 2x)$$
$$y_p = \text{Re}\, w_p = \tfrac{6}{65} \cos 2x + \tfrac{21}{130} \sin 2x \tag{47}$$

Alternatively, we could have written

$$8 + 14i = \sqrt{260}e^{i\theta} \qquad \theta = \arctan \tfrac{14}{8}$$

and

$$w_p = \frac{3e^{i2x}}{\sqrt{260}e^{i\theta}} = \frac{3}{\sqrt{260}} e^{i(2x-\theta)} \tag{48}$$

Therefore

$$y_p = \frac{3}{\sqrt{260}} \cos (2x - \theta) \tag{49}$$

The two expressions (47) and (49) for y_p are exactly equivalent.

Example 3 $y'' + 4y = 3 \sin 2x$

$$p(\lambda) = \lambda^2 + 4 = 0 \qquad \lambda = \pm 2i$$
$$y = \text{Im}\, w$$
$$w'' + 4w = 3e^{i2x}$$
$$w_p = \frac{3xe^{i2x}}{p'(2i)} = \frac{3xe^{i2x}}{4i} = -\tfrac{3}{4}ixe^{i2x}$$
$$y_p = \text{Im}\, w_p = -\tfrac{3}{4}x \cos 2x$$

An alternative way of solving $L(y) = k \cos \beta x$ or $L(y) = k \sin \beta x$ is to assume a solution of the form

$$y_p = A \cos \beta x + B \sin \beta x \tag{50}$$

and to determine the coefficients A and B by substituting into the differential equation and equating the coefficients of $\cos \beta x$ and $\sin \beta x$ on both sides of the equation. If $p(i\beta) = 0$, that is, if $\cos \beta x$ (and $\sin \beta x$) is a solution of the homogeneous equation, the form

$$y_p = x(A \cos \beta x + B \sin \beta x) \tag{51}$$

must be used in place of (50).

Example 4 $y'' + 7y' + 12y = 3 \cos 2x$

$$y_p = A \cos 2x + B \sin 2x$$

Substituting into the differential equation, we obtain

$$(8A + 14B) \cos 2x + (8B - 14A) \sin 2x = 3 \cos 2x$$

Therefore (why?)

$$(8A + 14B) = 3 \qquad 8B - 14A = 0$$

or

$$A = \tfrac{6}{65} \qquad B = \tfrac{21}{130}$$

and

$$y_p = \tfrac{6}{65} \cos 2x + \tfrac{21}{130} \sin 2x$$

III. $f(x) = B_0 + B_1x + \cdots + B_nx^n$ The differential equation

$$ay'' + by' + cy = B_0 + B_1x + \cdots + B_nx^n \tag{52}$$

clearly has a polynomial for a particular solution. If $p(0) = c \neq 0$, the substitution of

$$y_p = Q_n(x) = A_0 + A_1x + \cdots + A_nx^n \tag{53}$$

into the differential equation will yield a polynomial of degree n on the left-hand side of the equation. The coefficients A_k can then be obtained by equating the coefficients of like powers of x on both sides of the equation. If $p(0) = c = 0$ but $p'(0) = b \neq 0$, we must assume $y_p = xQ_n(x)$ in order to obtain a polynomial of degree n on the left side of the equation. Similarly, if $p(0) = p'(0) = 0$, we must assume $y_p = x^2Q_n(x)$.

Example 5 $y'' + 3y' = 2x^2 + 3x$

Assume

$$y_p = x(Ax^2 + Bx + C)$$

Substituting into the equation, we obtain after simplification

$$9Ax^2 + (6A + 6B)x + 2B + 3C = 2x^2 + 3x$$

and equating coefficients of different powers of x, we obtain

$$9A = 2 \qquad 6A + 6B = 3 \qquad 2B + 3C = 0$$

Therefore

$$A = \tfrac{2}{9} \qquad B = \tfrac{5}{18} \qquad C = -\tfrac{5}{27}$$

and the particular solution is

$$y_p = \tfrac{2}{9}x^3 + \tfrac{5}{18}x^2 - \tfrac{5}{27}x$$

IV. $f(x) = (B_0 + B_1x + \cdots + B_nx^n)e^{\alpha x}$ For the equation

$$ay'' + by' + cy = (B_0 + B_1x + \cdots + B_nx^n)e^{\alpha x} \tag{54}$$

we must assume

$$y_p = Q_n(x)e^{\alpha x} \qquad p(\alpha) \neq 0 \tag{55}$$

or

$$y_p = xQ_n(x)e^{\alpha x} \qquad p(\alpha) = 0, p'(\alpha) \neq 0 \tag{56}$$

or

$$y_p = x^2Q_n(x)e^{\alpha x} \qquad p(\alpha) = p'(\alpha) = 0 \tag{57}$$

By allowing α in these equations to be a complex number, we can also solve the equation

$$ay'' + by' + cy = (B_0 + B_1x + \cdots + B_nx^n)e^{\alpha x}\cos\beta x \tag{58}$$

or the same equation with $\cos\beta x$ replaced by $\sin\beta x$.

Example 6 $y'' + 4y = xe^x \sin x$

Let $y = \operatorname{Im} w$.

$$w'' + 4w = xe^xe^{ix} = xe^{(1+i)x}$$

Assume

$$w_p = (Ax + B)e^{(1+i)x}$$

Substituting, we obtain

$$(2i + 4)Ax + (4 + 2i)B + (2 + 2i)A = x$$

Therefore

$$A = \frac{1}{2i + 4} \qquad \text{and} \qquad B = -\frac{1 + i}{6 + 8i}$$

and

$$w_p = \frac{xe^xe^{ix}}{4 + 2i} - \frac{1 + i}{6 + 8i}e^xe^{ix}$$

$$y_p = \operatorname{Im} w_p = \tfrac{1}{10}xe^x(-\cos x + 2\sin x) - \tfrac{1}{50}e^x(-\cos x + 7\sin x)$$

Problems 5-5

Find particular solutions of the following equations:

1. $y'' + 3y' - 5y = 4e^{2x} + 6e^{-3x}$
2. $y'' + 3y' + 5y = 2\sin 3x$
3. $y'' + 9y = 4\cos 3x$
4. $y'' + 3y' - 4y = 2e^{-4x} + 5$
5. $y'' + 4y' + 4y = 3e^{-2x} + e^{-x}$
6. $y'' - 3y' + 4y = x^3 + 3x$
7. $y'' - 3y' + y = 3e^x \sin x$
8. $y'' - 3y' = 2x^2 + 3e^x$
9. $y'' + 2y' + 2y = 2e^x \cos x$
10. $y'' - 4y = 3xe^{2x}$

5-6 Variation of parameters

The method of variation of parameters enables us to find a particular solution of a nonhomogeneous equation whenever two linearly independent solutions of the homogeneous equation are known. This method works for any linear equation *even when the coefficients are not constant.* We consider the equation

$$a(x)y'' + b(x)y' + c(x)y = f(x) \tag{59}$$

where all the functions are continuous in some interval I and $a(x) \neq 0$ in I. We assume that $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the homogeneous equation. The general solution of the homogeneous equation is therefore

$$y_h = c_1 y_1(x) + c_2 y_2(x) \tag{60}$$

We now try to find a solution of the nonhomogeneous equation (59) by replacing the constants c_1 and c_2 by functions of x. Therefore we look for a solution in the form

$$y = v_1(x)y_1(x) + v_2(x)y_2(x) \tag{61}$$

By substituting this expression into the differential equation (59), we will get only one condition that the two functions $v_1(x)$ and $v_2(x)$ must satisfy. We therefore can impose another condition. This can be rather arbitrarily imposed but should be such as to simplify the determination of $v_1(x)$ and $v_2(x)$.

Let us calculate the derivatives of (61). We have

$$y' = v_1 y_1' + v_2 y_2' + (v_1' y_1 + v_2' y_2) \tag{62}$$

Before proceeding to calculate y'', we note that if we require

$$v_1' y_1 + v_2' y_2 = 0 \tag{63}$$

then no second derivatives of v_1 and v_2 will appear in y''. We therefore take (63) as one condition on v_1 and v_2. Calculating y'', with the condition (63), we obtain

$$y'' = v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2' \tag{64}$$

Substituting (61), (62), and (64) into the differential equation, we obtain

$$(ay_1'' + by_1' + cy_1)v_1 + (ay_2'' + by_2' + cy_2)v_2 + v_1' a y_1' + v_2' a y_2' = f(x) \tag{65}$$

The coefficients of v_1 and v_2 are zero because y_1 and y_2 are solutions of the homogeneous equation. Therefore we have the two equations

$$v_1' y_1 + v_2' y_2 = 0$$
$$v_1' y_1' + v_2' y_2' = \frac{f(x)}{a(x)} \tag{66}$$

to determine v_1 and v_2. These equations can be solved for v_1' and v_2' provided

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' \neq 0 \tag{67}$$

This determinant is the Wronskian of y_1 and y_2 and is always different from zero if y_1 and y_2 are linearly independent. Solving Eqs. (66), we obtain

$$v_1' = -\frac{y_2(x)f(x)}{a(x)W[y_1,y_2]} \qquad v_2' = \frac{y_1(x)f(x)}{a(x)W[y_1,y_2]}$$

Therefore

$$v_1 = -\int \frac{y_2(x)f(x)\,dx}{a(x)W[y_1,y_2]} \qquad v_2 = \int \frac{y_1(x)f(x)\,dx}{a(x)W[y_1,y_2]} \tag{68}$$

and a solution of the nonhomogeneous equation is

$$y = v_1(x)y_1(x) + v_2(x)y_2(x) \tag{69}$$

Example 1 Solve: $y'' + y = \sec x$

Linearly independent solutions of the homogeneous equation are

$$y_1 = \cos x \qquad y_2 = \sin x$$

and the general solution of the homogeneous equation is

$$y_h = c_1 \cos x + c_2 \sin x$$

We have

$$W[y_1,y_2] = y_1 y_2' - y_2 y_1' = \cos x\,(\cos x) - (\sin x)(-\sin x) = 1$$

Therefore

$$v_1 = -\int \sin x \sec x\,dx = -\int \frac{\sin x\,dx}{\cos x} = \ln|\cos x|$$
$$v_2 = \int \cos x \sec x\,dx = \int dx = x$$
$$y_p = \cos x \ln|\cos x| + x \sin x$$
$$y = y_h + y_p = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$$

Problems 5-6

Solve by variation of parameters:

1. $y'' + y = \tan x$
2. $y'' - 5y' + 6y = 2e^{4x}$
3. $y'' - y = \sin^2 x$
4. $y'' - y = x - 2e^{-x}$
5. $y'' + 4y' + 4y = \dfrac{e^{-2x}}{x^2}$
6. By variation of parameters show that the solution of

DE: $y'' + \lambda^2 y = f(x) \qquad \lambda > 0$

IC: $y(0) = 0$

$y'(0) = 0$

is

$$y = \frac{1}{\lambda}\int_0^x f(t) \sin \lambda(x - t)\,dt$$