Midterm I – MA 4335.

Due – 03/09/15 –

Consider the problem

3.0 $\begin{cases} U_{tt} = c^2 U_{xx} & x \in \mathbb{R},\ t > 0 \\ U(x,0) = 0, \quad U_t(x,0) = \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| \geq a \end{cases} \end{cases}$

for $c > 0$ and some $a > 0$.

i) Find analytically the solution $U(x,t)$.

ii) Let $a = 1$. For $c = 1$ and $c = 100$ plot the profile of U at succesive time instants
$t = \frac{a}{2c}, \frac{a}{c}, \frac{3a}{2c}, \frac{2a}{c}, \frac{5a}{c}$

Perform also a surface plot of $U(x,t)$ as a function of (x,t). Discuss the results and compare the two situations $c = 1$ and $c = 10$.

iii) Find $\max_{x \in \mathbb{R}} U(x,t)$ as a function of t

and prove that the maximum principle for the wave equation does not hold!

2. Prove the uniqueness of the following problem by using the energy method.

20p

$$\begin{cases} U_t = kU_{xx} & 0 < x < L, \ t > 0 \\ U(x,0) = \phi(x), \ U_x(0,t) = g(t), \ U_x(L,t) = h(t) \end{cases}$$

where g and h are smooth function of time and ϕ is a smooth function of space.

3. Consider the following problem

30p

$$\begin{cases} U_t - kU_{xx} + bU = 0 & x \in \mathbb{R}, \ t > 0 \\ U(x,0) = \phi(x) \end{cases}$$

and $b > 0$ constant

i) Solve the problem to find $U(x,t)$
(Hint: Use change of variable $V(x,t) = e^{bt} U(x,t)$).

ii) For $b = 1$ and $b = 10$ and for $k = 1$ with

$$\phi(x) = \begin{cases} 1 & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

plot the solution on a surface plot.
Compare and discuss the results.

iii) Discuss numerically what happens when k gets larger or very small. For this part fix $b=1$ and consider

$$\phi(x) = \begin{cases} 1 & |x|<1 \\ 0 & |x| \geq 1. \end{cases}$$

4\. Solve

20p
$$\begin{cases} U_{xx} - 3U_{xt} - 4U_{tt} = 0 \\ U(x,0) = x^2, \quad U_t(x,0) = e^x \end{cases}$$

and plot the solution $U(x,t)$ numerically on a surface plot.

(Hint: Factor the operator as we did for the wave equation)

Problem 1

i) $$U(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy \quad \text{where}$$ 5p

ψ is the initial momentum $\psi(y) = \begin{cases} 1 & |y|<a \\ 0 & |y|\geq a \end{cases}$

Observe that

$$\psi(y) = H(y+a) - H(y-a) \quad \text{for } y \neq \{a,-a\}$$

H is the Heaviside function $H(z) = \begin{cases} 1 & z>0 \\ 0 & z<0 \end{cases}$

Thus

$$U(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \left[H(y+a) - H(y-a)\right] dy$$

5p

$$= \frac{1}{2c}\left\{\text{length of } (x-ct,\ x+ct)\cap(-a,a)\right\}$$

(ii) Plot of solution profile at $t \in \left\{\frac{a}{2c}, \frac{a}{c}, \frac{3a}{2c}, \frac{2a}{c}, \frac{5a}{c}\right\}$

For $c=1$ 2p

For $c=100$ 2p

Plot of surface plot of solution as a function of (x,t)

For $c=1$ 2p

For $c=100$ 2p

Comparison between the solution with

$c = 1$, $c = 100$. 2p.

iii) Observe first that

For $x + ct < -a$ or $x - ct > a$ we have $u(x,t) = 0$. 1p

Then, let us focus on the case

$$x \in [-ct - a, ct + a]$$

We identify four situations.

I. $\left.\begin{array}{l} x - ct < -a \\ x + ct < a \end{array}\right\} \Rightarrow u(x,t) = \frac{1}{2c} \text{length}(-a, x+ct)$

$= \frac{1}{2c}(x + ct + a)$ 1p

II. $\left.\begin{array}{l} x - ct < -a \\ x + ct > a \end{array}\right\} \Rightarrow u(x,t) = \frac{1}{2c} \text{length}(-a, a) = \frac{a}{c}$

1p

III. $\left.\begin{array}{l} x - ct > -a \\ x + ct < a \end{array}\right\} \Rightarrow u(x,t) = \frac{1}{2c} \text{length}(x-ct, x+ct)$

$= t$ 1p

IV. $\left.\begin{array}{l} x - ct > -a \\ x + ct > a \end{array}\right\} \Rightarrow u(x,t) = \frac{1}{2c} \text{length}(x-ct, a)$

$= \frac{a - x + ct}{2c}$ 1p

We can conclude that

$$\max_x U(x,t) = \begin{cases} \frac{a}{c} & \text{if } a > ct \\ t & \text{if } a < ct. \end{cases}$$ 3p

From the above discussion we can see that $\max_x U(x,t)$ is attained for

$$x \in \left(\min\{ct-a, a-ct\}, \max\{ct-a, a-ct\} \right)$$

This contradicts the maximum principle which will predict a maximum of $U(x,t)$ attained only for $t=0$!! Thus maximum principle is not true for the wave equation. 2p.

Problem 2

Assume by contradiction that there are two solution of the problem U_1, U_2. 5p.

Consider $W = U_1 - U_2$. W solves.

$$(2)\begin{cases} W_t = k\, W_{xx} & 0 < x < L,\ t > 0 \\ W(x,0) = 0 \\ W_x(0,t) = W_x(L,t) = 0 \end{cases}$$

Multiply equation (2) with w and integrate from 0 to L with respect to x 5p

$$\int_0^L W_t W = k \int_0^L W_{xx} W = k \int_0^L (W_x W)_x - k \int_0^L (W_x)^2$$

$$\frac{1}{2} \frac{d}{dt}\left(\int_0^L W^2\right) = \underbrace{k\, W_x W \Big|_0^L}_{=0} - \underbrace{k \int_0^L (W_x)^2}_{\leq 0}$$

Thus $\dfrac{d}{dt}\left(\int_0^L W^2\right) \leq 0$ 5p

Then we have

$$\int_0^L w^2(x,t)\,dx \leq \int_0^L w^2(x,0)\,dx = 0$$

$$\Downarrow$$

$$w(x,t) = 0 \,. \quad \forall t$$

Contradiction with the hypotesis Q.e.d.

Problem 3

i) Let $V(x,t) = e^{bt} U(x,t)$

$$\left.\begin{aligned} V_t &= b e^{bt} U + e^{bt} U_t \\ V_{xx} &= e^{bt} U_{xx} \end{aligned}\right\} \quad (*) \quad 2p$$

observe that $(*)$ imply

$$\begin{cases} V_t - k V_{xx} = 0 & , x \in \mathbb{R}, t > 0 \\ V(x,0) = U(x,0) = \phi(x) \end{cases} \quad 2p$$

$$V(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\, dy \quad 1p$$

Thus

$$U(x,t) = \frac{e^{-bt}}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\, dy \quad 2p$$

ii) Plot of surface profile for solution $U(x,t)$

For $b=1$ 5p

For $b=10$ 5p

Comparison 3p

iii) Discussion with respect to k large or small

10p

Problem 4

Observe that

$$U_{xx} - 3U_{xt} - 4U_{tt} = (\partial_x - 4\partial_t)(\partial_x + \partial_t)U$$ 3p

Let $V = (\partial_x + \partial_t)U = U_x + U_t$

Then $U_{xx} - 3U_{xt} - 4U_{tt} = 0$ is equivalent

to

$$(1) \quad \begin{cases} V_x - 4V_t = 0 \\ U_x + U_t = V \end{cases}$$ 2p

$V_x - 4V_t = 0 \Rightarrow V(x,t) = f(4x+t)$ (2) for some function f. $f \in C^1$

1p

Then U solves

$$U_x + U_t = f(4x+t) \quad (3)$$

$$U(x,t) = g(x-t) + U_p(x,t) \quad (4)$$ 2p

where g is a C^1 function and U_p is a particular solution of (3)

Let us try $U_p(x,t) = h(4x+t)$ 1p.

Then
$$\partial_x U_p + \partial_t U_p = 4h'(4x+t) + h'(4x+t) =$$ 1p.
$$= 5h'(4x+t)$$

Thus if we take $h(s) = \frac{1}{5}\int_0^s f(y)\,dy$

we have that $U_p(x,t) = h(4x+t)$ satisfies.

$$\frac{\partial}{\partial x}U_p + \frac{\partial}{\partial t}U_p = f(4x+t) \quad (5)$$ 2p.

From (5) and (4) we obtain

$$U(x,t) = g(x-t) + \frac{1}{5}\int_0^{4x+t} f(y)\,dy \quad (5*)$$ 2p

Next we determine g and f from the initial data

$$U(x,0) = x^2 \Rightarrow g(x) + \frac{1}{5}\int_0^{4x} f = x^2 \quad (6)$$ 1p

$$U_t(x,0) = e^x \Rightarrow -g'(x) + \frac{f(4x)}{5} = e^x \quad (7)$$

Differentiate the first equation (6) and add it to (7) to obtain.

$$\frac{4f(4x)}{5} + \frac{f(4x)}{5} = 2x + e^x$$

$$\Rightarrow f(4x) = 2x + e^x \Rightarrow f(x) = \frac{x}{2} + e^{\frac{x}{4}} \quad (8)$$

1p

From (6) get

$$g(x) = x^2 - \frac{1}{5}\int_0^{4x} \left(\frac{y}{2} + e^{\frac{y}{4}}\right) dy \qquad 1p$$

$$g(x) = x^2 - \frac{1}{5}\left(\frac{y^2}{4} + 4e^{\frac{y}{4}}\right)\Big|_0^{4x} =$$

$$g(x) = x^2 - \frac{4x^2}{5} - \frac{4e^x}{5} + \frac{4}{5} =$$

$$= \frac{x^2}{5} - \frac{4e^x}{5} + \frac{4}{5} \quad (9) \qquad 1p$$

Thus the solution $U(x,t)$ is given by using (8) and (9) in (5*)

$$U(x,t) = \frac{(x-t)^2}{5} - \frac{4e^{x-t}}{5} + \frac{4}{5} + \frac{1}{5}\int_0^{4x+t}\left(\frac{y}{2} + e^{\frac{y}{4}}\right) dy$$

$$U(x,t) = \frac{(x-t)^2}{5} - \frac{4e^{x-t}}{5} + \frac{4}{5} + \frac{1}{5}\left(\frac{y^2}{4} + 4e^{\frac{y}{4}}\right)\Bigg|_0^{4x+t}$$

$$= \frac{(x-t)^2}{5} - \frac{4e^{x-t}}{5} + \cancel{\frac{4}{5}} + \frac{(4x+t)^2}{20} + \frac{4}{5}e^{x+\frac{t}{4}} - \cancel{\frac{4}{5}}$$

$$= \frac{(x-t)^2}{5} - \frac{4e^{x-t}}{5} + \frac{(4x+t)^2}{20} + \frac{4}{5}e^{x+\frac{t}{4}}$$

P.