Can the dimension of a fractal set or measure be inferred from finite-dimensional projections?

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Outline

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Background

Hilbert space case

Banach space case

Primary question

$\mathfrak{B}$ – Banach space
$A \subset \mathfrak{B}$ – compact set of interest

QUESTION – For a typical projection $f : \mathfrak{B} \to \mathbb{R}^m$, do we have

$$\dim(f(A)) = \dim(A)?$$

MOTIVATION

- A – invariant set for an infinite-dimensional dynamical system
- Can we infer $\dim(A)$ from finite-dimensional data?

NOTE – f Lipschitz on $A \implies \dim(f(A)) \leqslant \dim(A)$

Finite-dimensional Banach spaces

$\mathfrak{B} = \mathbb{R}^n$
$\dim_H$ – Hausdorff dimension

Theorem (Sauer/Yorke)

Let $A \subset \mathbb{R}^n$ be compact. For a prevalent $\mathcal{C}^1$ projection $f : \mathbb{R}^n \to \mathbb{R}^m$,

$$\dim_H(f(A)) = \min \{m, \dim_H(A)\}.$$

Dimension estimates via embeddings

$\mathfrak{H}$ – real Hilbert space
$A \subset \mathfrak{H}$ – compact set
d – box-counting dimension of A
$m \in \mathbb{Z}$ – dimension of target space

Theorem (Hunt/Kaloshin)

For every α satisfying

$$0 < \alpha < \left(\frac{m - 2d}{m}\right)\left(\frac{1}{1 + d/2}\right),$$

a prevalent $\mathcal{C}^1$ projection $f : \mathfrak{H} \to \mathbb{R}^m$ satisfies

$$C|f(x) - f(y)|^\alpha \geqslant |x - y|$$

on A.

Dimension estimates via embeddings

Corollary (Hunt/Kaloshin)

For a prevalent $\mathcal{C}^1$ projection $f : \mathfrak{H} \to \mathbb{R}^m$,

$$\left(\frac{m-2d}{m}\right)\left(\frac{1}{1+d/2}\right)\dim_H(A) \leqslant \dim_H(f(A)) \leqslant \dim_H(A).$$

QUESTIONS

1. Can the factor $(m-2d)/m$ be removed?
2. Can the intrinsic factor $1/(1+d/2)$ be improved?

Outline

Thickness exponent

$\mathfrak{H}$ – real Hilbert space
$A \subset \mathfrak{H}$ – compact set

Definition

The thickness exponent of A is the scaling exponent

$$\tau(A) = \limsup_{\varepsilon \to 0} \frac{\log(d(A, \varepsilon))}{\log(1/\varepsilon)},$$

where $d(A, \varepsilon)$ is the minimal dimension of finite-dimensional subspaces of $\mathfrak{H}$ that ε-approximate A.

Thickness exponent: Intuition

$$\tau(A) = \limsup_{\varepsilon \to 0} \frac{\log(d(A, \varepsilon))}{\log(1/\varepsilon)},$$

- Finite-dimensional disks
  - Thickness exponent zero
  - Arbitrarily high Hausdorff dimension
- $\{0, e_2/\log(2), e_3/\log(3), \ldots\}$
  - Hausdorff dimension zero
  - $\infty =$ thickness exponent
- Any A: $\tau(A) \leqslant \dim_B(A)$

Hilbert space result

$\mathfrak{H}$ – real Hilbert space
$A \subset \mathfrak{H}$ – compact set

Theorem (Ott/Hunt/Kaloshin)

For a prevalent $\mathcal{C}^1$ *projection* $f : \mathfrak{H} \to \mathbb{R}^m$,

$$\dim_H(f(A)) \geqslant \min\left\{m, \frac{\dim_H(A)}{1 + \tau(A)/2}\right\}.$$

- This is sharp!
- Dimension preservation result when $\tau(A) = 0$

Non-computable attractors?

Ott/Hunt/Kaloshin estimate –

$$\dim_H(f(A)) \geqslant \min\left\{m, \frac{\dim_H(A)}{1+\tau(A)/2}\right\}.$$

QUESTION Does there exist a natural attractor A with $\tau(A) > 0$ and $0 < \dim_H(A) < \infty$?

- $2D$ Naiver-Stokes attractor
  - Finite Hausdorff dimension
  - Thickness exponent zero (Robinson)

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Background

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Banach space case

A question of Robinson

QUESTION Does an Ott/Hunt/Kaloshin result hold for Banach spaces?

ISSUE

- $\mathfrak{B}$ – Banach space
- F – finite-dimensional subspace of $\mathfrak{B}$
- How does the unit ball in F^* embed into $\mathfrak{B}^*$?

Dual thickness exponent

$\mathfrak{B}$ – Banach space
$A \subset \mathfrak{B}$ – compact set

Definition

For $\theta > 0$, define the scaling exponent

$$\tau^*_\theta(A) = \limsup_{\varepsilon \to 0} \frac{\log(n_\theta(A, \varepsilon))}{\log(1/\varepsilon)},$$

where $n_\theta(A, \varepsilon)$ is the minimal dimension of linear subspaces $V \subset \mathfrak{B}^*$ such that if $x, y \in A$ satisfy $|x - y| \geqslant \varepsilon$, then there exists $\psi \in V$ with $\|\psi\| = 1$ and

$$|\psi(x - y)| \geqslant \varepsilon^{1+\theta}.$$

Dual thickness exponent $\longrightarrow \tau^*(A) = \lim_{\theta \to 0} \tau^*_\theta(A)$

First Banach space result

$\mathfrak{B}$ – Banach space

$A \subset \mathfrak{B}$ – compact set

Theorem (Ott/Stout/Zhou)

For a prevalent $\mathcal{C}^1$ projection $f : \mathfrak{B} \to \mathbb{R}^m$,

$$\dim_H(f(A)) \geqslant \min\left\{m, \frac{\dim_H(A)}{1+\tau^*(A)}\right\}.$$

What about an estimate in terms of $\tau(A)$?

Thickness vs. dual thickness

Robinson –

$$\tau(A) = 0 \Longrightarrow \tau^*(A) = 0$$

Ott/Stout/Zhou –

$$\tau^*(A) \leqslant \frac{\tau(A)}{1 - \beta\tau(A)}$$

β – projection constant scaling factor

Second Banach space result

$\mathfrak{B}$ – Banach space
$A \subset \mathfrak{B}$ – compact set

Theorem (Ott/Stout/Zhou)

For a prevalent $\mathcal{C}^1$ projection $f : \mathfrak{B} \to \mathbb{R}^m$,

$$\dim_H(f(A)) \geqslant \min\left\{m, \frac{1-\beta\tau(A)}{1-\beta\tau(A)+\tau(A)}\dim_H(A)\right\}.$$

Requires $\beta\tau(A) < 1$!