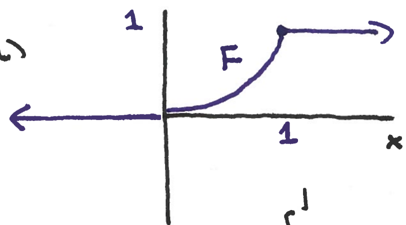
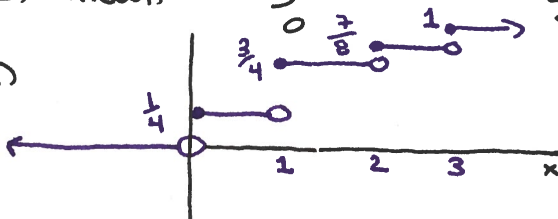
E1.2.3 (a)



(b) $f(x) = F'(x) = \begin{cases} 0, & \text{if } x \le 0; \\ 3x^2, & \text{if } 0 < x < 1; \\ 0, & \text{if } x > 1. \end{cases}$

(c) $\text{mean} = \int_0^1 x f(x)\,dx = \int_0^1 x(3x^2)\,dx = \int_0^1 3x^3\,dx = \frac{3}{4}x^4 \Big|_0^1 = \boxed{\frac{3}{4}}.$

E1.2.4 (a)



(b) $E[Z] = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{1}{8} = \boxed{9/8}$

(c) $Var[Z] = E[Z^2] - E[Z]^2 = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 4 \cdot \frac{1}{8} + 9 \cdot \frac{1}{8} - \left(\frac{9}{8}\right)^2 = 17/8 - \frac{81}{64} = \boxed{\frac{55}{64}}.$

E1.2.7 (a) Notice that the antiderivative of Rx^{R-1} is $x^R + C$.

Thus $F_X(x) = \begin{cases} 0, & x \le 0; \\ x^R, & \text{if } 0 \le x \le 1; \\ 1, & \text{if } x \ge 1. \end{cases}$

since $\int_0^x R t^{R-1}\,dt = t^R \Big|_0^x = x^R.$

(b) $E[X] = \int_0^1 (x)(Rx^{R-1})\,dx = \int_0^1 Rx^R\,dx = \frac{R\,x^{R+1}}{R+1}\Big|_0^1 = \boxed{R/R+1}$

(c) $Var[X] = E[X^2] - (R/R+1)^2 = \int_0^1 (x^2)(Rx^{R-1})\,dx - (R/R+1)^2 = \frac{R\,x^{R+2}}{R+2}\Big|_0^1 - (R/R+1)^2$

$= \boxed{R/(R+2) - \left(R/(R+1)\right)^2}$

P1.2.1 Since $N = 1_{A_1} + 1_{A_2} + \cdots + 1_{A_{13}}$, we have

$E[N] = E[1_{A_1}] + E[1_{A_2}] + \cdots + E[1_{A_{13}}].$

Next, for any k we have $E[1_{A_k}] = 1 \cdot \frac{1}{13} + 0 \cdot 12/13 = \frac{1}{13}.$

This is so because $P(A_k) = \frac{1}{13}$, and $1_{A_k} = \begin{cases} 1, & \text{if } A_k \text{ occurs}; \\ 0, & \text{otherwise}. \end{cases}$

Thus $E[N] = \frac{1}{13} + \frac{1}{13} + \cdots \frac{1}{13} = \boxed{1}.$

P1.2.4 First notice that N can take only the values $2, 3, 4, \ldots$

Let p denote the probability mass function. We have $p(2) = \frac{1}{2}$, since the game ends in exactly two tosses if the first two tosses are HH or TT, while there are four possibilities for the first two tosses (HH, TT, TH, HT).

For $p(3)$, we have $p(3) = \frac{2}{8} = \frac{1}{4}$, since only the configurations HTT and THH end the game in exactly three tosses, while there are 8 possibilities for the first 3 tosses. For general i in $\{2,3,4,\dots\}$, $p(i) = \frac{2}{2^i} = \frac{1}{2^{i-1}}$. This is so because only HTH$\cdots(**)$ and THT$\cdots(**)$ end the game in exactly i tosses, while there are 2^i possibilities for the first i tosses.

(b) $P(A) = \sum_{k=1}^{\infty} p(2k) = \sum_{k=1}^{\infty} \frac{1}{2^{2k-1}} = 2\sum_{k=1}^{\infty} \frac{1}{4^k} = 2\cdot\left[\frac{\frac{1}{4}}{1-\frac{1}{4}}\right] = \boxed{\frac{2}{3}}$.

$P(B) = p(2)+p(3)+p(4)+p(5)+p(6) = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32} = \boxed{\frac{31}{32}}$.

$P(A\cap B) = p(2)+p(4)+p(6) = \frac{1}{2}+\frac{1}{8}+\frac{1}{32} = \boxed{\frac{21}{32}}$.

$\boxed{\text{P1.2.6}}$ We represent the outcomes of a single roll of the dice in a matrix form:

$$\begin{pmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{pmatrix}.$$

Since tosses with equal die values are discarded, we ignore the diagonal of the matrix. The probability mass function for the sum is given by $p(3) = \frac{2}{30}$, $p(4) = \frac{2}{30}$, $p(5) = \frac{4}{30}$, $p(6) = \frac{4}{30}$, $p(7) = \frac{6}{30}$, $p(8) = \frac{4}{30}$, $p(9) = \frac{4}{30}$, $p(10) = \frac{2}{30}$, $p(11) = \frac{2}{30}$.

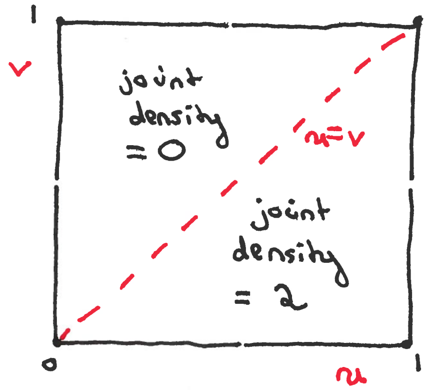
$\boxed{\text{P1.2.8}}$ • $E[Y] = E[a+bX] = E[a] + b[E[X]] = a + b\mu$.

• $\mathrm{Var}[Y] = E[Y^2] - (E[Y]^2)$
$= E[(a+bX)^2] - [(a+b\mu)^2]$
$= E[a^2] + 2ab[E[X]] + b^2E[X^2] - (a^2+2ab\mu+b^2\mu^2)$
$= a^2 + 2ab\mu + b^2E[X^2] - a^2 - 2ab\mu - b^2\mu^2$
$= b^2(E[X^2] - \mu^2)$
$= b^2\mathrm{Var}[X] = b^2\sigma^2$.

$\boxed{\text{P1.2.10}}$ Z can take values in $\{1,\dots,6\}$. The pmf p of Z is a convolution.

$p(1) = \Pr(X=0 \text{ and } Y=1) = p_X(0)\cdot p_Y(1) = (\frac{1}{2})(\frac{1}{6}) = \frac{1}{12}$;
$p(2) = \Pr(X=0 \text{ and } Y=2) = p_X(0)\cdot p_Y(2) = (\frac{1}{2})(\frac{1}{3}) = \frac{1}{6}$;
$p(3) = (\frac{1}{2})(\frac{1}{2}) = \frac{1}{4}$; $p(4) = (\frac{1}{2})(\frac{1}{6}) = \frac{1}{12}$;
$p(5) = (\frac{1}{2})(\frac{1}{3}) = \frac{1}{6}$; $p(6) = (\frac{1}{2})(\frac{1}{2}) = \frac{1}{4}$.

P1.2.13 Let $U = \max\{X, Y\}$ and $V = \min\{X, Y\}$. We will compute the joint distribution function F of U and V and then differentiate to find the joint density function.



[Case 1 -- $u \le v$]

$$\begin{aligned}
F(u,v) &= \Pr(U \le u \text{ and } V \le v) \\
&= \Pr(\max\{X,Y\} \le u \text{ and } \min\{X,Y\} \le v\} \\
&= \Pr(\max\{X,Y\} \le u) \\
&= \Pr(X \le u \text{ and } Y \le u) \\
&= \Pr(X \le u) \cdot \Pr(Y \le u) \quad \text{[using independence)} \\
&= u^2 .
\end{aligned}$$

[Case 2 -- $u > v$]

$$\begin{aligned}
F(u,v) &= \Pr(\min\{X,Y\} \le v \text{ and } \max\{X,Y\} \le u) \\
&= \Pr([X \le v \text{ and } Y \le u]) \\
&\quad + \Pr(Y \le v \text{ and } X \le u) \\
&\quad - PR(X \le v \text{ and } Y \le v) \\
&= vu + vu - v^2 = 2vu - v^2 \quad \text{[using independence]}
\end{aligned}$$

Call f the joint density function of U and V.

Case 1: If $0 < u < v < 1$, then

$$f(u,v) = \frac{\partial^2 F}{\partial v \partial u}(u,v) = \frac{\partial^2}{\partial v \partial u}[u^2] = \frac{\partial}{\partial v}[2u] = 0.$$

Case 2: If $0 < v < u < 1$, then

$$f(u,v) = \frac{\partial^2}{\partial v \partial u}[2vu - v^2] = \partial/\partial v\,[2v] = 2.$$