E 1.3.2 Let X be the number of defective items.

- $Pr(X = 1) = \binom{10}{1}(.05)^1(.95)^9$
- $Pr(X \leq 1) = \binom{10}{1}(.05)^1(.95)^9 + (.95)^{10}$

E 1.3.4

- $Pr(X = 2) = \frac{\lambda^2}{2!}e^{-\lambda} = \frac{2}{e^2}$
- $Pr(X \leq 2) = \frac{\lambda^2}{2!}e^{-\lambda} + \frac{\lambda^1}{1!}e^{-\lambda} + \frac{\lambda^0}{0!}e^{-\lambda} = \frac{2}{e^2} + \frac{2}{e^2} + \frac{1}{e^2} = \frac{5}{e^2}.$

P1.3.3

$$Pr(X \text{ is odd}) = \sum_{\substack{k=0 \\ k \text{ odd}}}^{\infty} Pr(X = k) = \sum_{j=0}^{\infty} Pr(X = 2j+1)$$

$$= \sum_{j=0}^{\infty} \frac{\lambda^{2j+1}}{(2j+1)!}e^{-\lambda}$$

$$= e^{-\lambda}\left[\frac{e^{\lambda} - e^{-\lambda}}{2}\right] = \frac{1 - e^{-2\lambda}}{2}.$$

Here I am using the fact that

$$(A) \quad \frac{e^{\lambda} - e^{-\lambda}}{2} = \sum_{\substack{k=0 \\ k \text{ odd}}}^{\infty} \lambda^k/k!, \quad (B) \quad \frac{e^{\lambda} + e^{-\lambda}}{2} = \sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} \lambda^k/k!.$$

These formulas follow from $e^{\lambda} = \sum_{k=0}^{\infty} \lambda^k/k!$, $e^{-\lambda} = \sum_{k=0}^{\infty} (-\lambda)^k/k!$.

P1.3.4

$$E[V] = \sum_{n=0}^{\infty}\left(\frac{1}{1+n}\right)\left(\frac{\mu^n}{n!}e^{-\mu}\right)$$

$$= e^{-\mu}\sum_{n=0}^{\infty}\frac{\mu^n}{(n+1)!} = \frac{e^{-\mu}}{\mu}\sum_{n=0}^{\infty}\frac{\mu^{n+1}}{(n+1)!}$$

$$= \frac{e^{-\mu}}{\mu}\left[e^{\mu} - 1\right].$$

P1.3.5 For Y, we need not compute the distribution explicitly. Since X is binomial, we have

$$E[X] = Np$$

$$Var[X] = E[X^2] - (E[X])^2 = Np(1-p),$$

so $$E[X^2] = Np(1-p) + N^2p^2.$$

$\rightarrow$ $$E[XY] = E[X(N-X)] = E[NX - X^2]$$

$$= N\,E[X] - E[X^2]$$

$$= N(Np) - [Np(1-p) + N^2p^2]$$

$\rightarrow$ $$Cov[X,Y] = E[XY] - E[X]E[Y]$$

$$= [N^2p - Np + Np^2 - N^2p^2] - (Np)(N - Np)$$

$$= Np[p-1] \quad \square$$

P1.3.9 $p_Z(n)$ for $n = 0,1,2,\dots$, the mass function for Z, is given by

$$p_Z(n) = \sum_{k=0}^{n} p_X(k)\, p_Y(n-k)$$

$$= \sum_{k=0}^{n} (1-\pi)\pi^k (1-\pi)\pi^{n-k}$$

$$= \sum_{k=0}^{n} (1-\pi)^2 \pi^n$$

$$= \left(\sum_{k=0}^{n} 1\right)\left[(1-\pi)^2\pi^n\right] = (n+1)(1-\pi)^2\pi^n .$$

This is indeed a negative binomial. Z is counting the number of failures before the second success.

P1.3.13 Pr(sample of 10 gives 2 or more defective parts)

$$= 1 - \left[\binom{10}{0}(.05)^0(.95)^{10} + \binom{10}{1}(.05)^1(.95)^9 \right].$$

We assume that the sample of 10 is representative and that each part is independent of the others.

E1.4.1 Let X denote lightbulb lifetime.

- $\Pr(X > 1.5\text{ yr}) = e^{-(2)(1.5)} = e^{-3}$
- $\Pr(X = 1.5\text{ yr}) = 0$

E1.4.2 Let $\hat{m}$ denote the median. We have $\int_0^{\hat{m}} \lambda e^{-\lambda t}\,dt = \frac{1}{2}$, so

$$\left(-e^{-\lambda t}\right)\Big|_0^{\hat{m}} = \frac{1}{2}, \text{ or } \left(-e^{-\lambda \hat{m}}\right) + 1 = \frac{1}{2}.$$

Solving for $\hat{m}$ yields $\hat{m} = \ln(2)/\lambda$. The mean is $\frac{1}{\lambda}$, so median < mean in this case.

E1.4.6a The density function of Y is computed by change of variable. Using Eq. (1.15) of Pinsky/Karlin, we have

$$f_Y(y) = \frac{1}{g'(u)} f_U(u) \qquad (\text{where } y = g(u) = -\ln(1-u))$$

$$= (1-u)\cdot 1$$

$$= 1 - [1 - e^{-y}] = e^{-y}.$$

Thus Y is exponential w/ parameter $\lambda = 1$!

E1.4.7 By convolution, R has density

$$f_R(r) = \int_0^r \lambda e^{-\lambda(r-s)} \lambda e^{-\lambda s}\,ds$$

$$= \lambda^2 \int_0^r e^{-\lambda r}\,ds$$

$$= \lambda^2 e^{-\lambda r}\int_0^r ds = \lambda^2 e^{-\lambda r}\left(s\Big|_0^r\right) = \lambda^2 r e^{-\lambda r}.$$

This is a gamma distribution.

P1.4.3 For an interval $[a,b]$, we will write $\mathbb{1}_{[a,b]}(x) = \begin{cases} 1, & \text{if } a \le x \le b; \\ 0, & \text{o.w.} \end{cases}$ for the indicator function of $[a,b]$. The density functions for X and $(-Y)$ are $\mathbb{1}_{[\theta-\frac{1}{2},\,\theta+\frac{1}{2}]}$ and $\mathbb{1}_{[-\theta-\frac{1}{2},\,-\theta+\frac{1}{2}]}$, respectively.

The density function for $W = X - Y$ is given by convolution:

$$f_W(w) = \int_{-\infty}^{\infty} \mathbb{1}_{[\theta-\frac{1}{2},\,\theta+\frac{1}{2}]}(y)\, \mathbb{1}_{[-\theta-\frac{1}{2},\,-\theta+\frac{1}{2}]}(w-y)\,dy$$

$$= \int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} \mathbb{1}_{[-\theta-\frac{1}{2},\,-\theta+\frac{1}{2}]}(w-y)\,dy$$

(let $t = w-y$)

$$= -\int_{w-[\theta-\frac{1}{2}]}^{w-[\theta+\frac{1}{2}]} \mathbb{1}_{[-\theta-\frac{1}{2},\,-\theta+\frac{1}{2}]}(t)\,dt$$

$$= \int_{w-[\theta+\frac{1}{2}]}^{w-[\theta-\frac{1}{2}]} \mathbb{1}_{[-\theta-\frac{1}{2},\,-\theta+\frac{1}{2}]}(t)\,dt$$

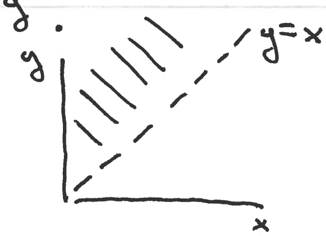
$$= \text{Length}\left([w-(\theta+\tfrac{1}{2}),\, w-(\theta-\tfrac{1}{2})] \cap [-\theta-\tfrac{1}{2},\, -\theta+\tfrac{1}{2}] \right)$$

$$= \text{Length}\left([w,\, w+1] \cap [0,1] \right)$$

$$= (1-|w|)\,\mathbb{1}_{[-1,1]}(w)$$

$$= \begin{cases} 1+w, & \text{for } -1 \le w < 0; \\ 1-w, & \text{for } 0 \le w \le 1; \\ 0, & \text{for } |w| > 1. \end{cases}$$

P1.4.5 Since the RVs are independent, the density for X, Y viewed jointly is the product $f(x,y) = 2e^{-2x} \cdot 3e^{-3y} = 6e^{-2x}e^{-3y}$.



$$\Pr(X < Y) = \int_0^\infty \int_x^\infty 6e^{-2x}e^{-3y}\,dy\,dx$$

$$= 6\int_0^\infty e^{-2x} \int_x^\infty e^{-3y}\,dy\,dx$$

$$= 6\int_0^\infty e^{-2x}\left(\frac{e^{-3x}}{3} \right) dx$$

$$= 2\int_0^\infty e^{-5x}\,dx$$

$$= 2/5.$$