E1.5.2 Let's assume (without loss of generality) that the first chip is red. Let X be the number of draws required to see the red chip again. Using the tail formula (1.49), we have

$$E[X] = \Pr(X\geq 1) + \Pr(X\geq 2) + \Pr(X \geq 3) + \Pr(X\geq 4)$$
$$= 1 + \left(\tfrac{3}{4}\right) + \left(\tfrac{3}{4}\right)\left(\tfrac{2}{3}\right) + \left(\tfrac{3}{4}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{1}{2}\right)$$
$$= \boxed{5/2}.$$

E1.5.3 X has density function $f(t) = \begin{cases} \lambda e^{-\lambda t}, & t\geq 0; \\ 0, & t<0. \end{cases}$

(a) $E[X] = \int_0^\infty t(\lambda e^{-\lambda t})\,dt = \lambda \int_0^\infty t e^{-\lambda t}\,dt$ $\left[\begin{array}{ll} u=t & dv = e^{-\lambda t}dt \\ du=dt & v = -\frac{e^{-\lambda t}}{\lambda} \end{array}\right]$

$$= \lambda\left(\left[-\frac{t}{\lambda}e^{-\lambda t}\right]\Big|_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-\lambda t}\,dt\right)$$
$$= \lambda\left([0]-[0] - \left[\frac{1}{\lambda^2}e^{-\lambda t}\Big|_0^\infty\right]\right) = \lambda\left[0-\frac{1}{\lambda^2}\right] = \boxed{\frac{\lambda}{\lambda^2}=\frac{1}{\lambda}}$$

(b) $E[X] = \int_0^\infty [1-F(t)]\,dt = \int_0^\infty e^{-\lambda t}\,dt = -\frac{e^{-\lambda t}}{\lambda}\Big|_0^\infty = \boxed{\frac{1}{\lambda}}.$

The method in (b) is easier!

P1.5.2 Let's compute $1-F_Z(z) = \Pr(Z>z)$ first. If $z<0$, then $\Pr(Z>z)=1$. If $z\geq 0$, then $\Pr(Z>z) =$

$$= \Pr(\min\{X_1,\dots,X_n\} > z)$$
$$= \Pr(X_1 > z \text{ and } X_2 > z \text{ and} \cdots \text{and } X_n > z)$$
$$= \Pr(X_1>z)\cdot \Pr(X_2>z)\cdot \cdots \cdot \Pr(X_n>z) \quad \text{(independence)}$$
$$= e^{-\lambda z}\cdot e^{-\lambda z}\cdot \cdots \cdot e^{-\lambda z} = e^{-\lambda n z}$$

Thus $1-F_Z(z) = \begin{cases} 1, & \text{if } z<0; \\ e^{-\lambda n z}, & \text{if } z\geq 0 \end{cases}$, so $F_Z(z) = \begin{cases} 0, & \text{if } z<0; \\ 1-e^{-\lambda n z}, & \text{if } z\geq 0. \end{cases}$

P1.5.3 (a) $\Pr(X>k) = \sum_{j=k+1}^\infty p(j) = \sum_{j=k+1}^\infty p(1-p)^j = p(1-p)^{k+1}\sum_{j=k+1}^\infty (1-p)^{j-(k+1)}$

$$= p(1-p)^{k+1}\big/1-[1-p] = (1-p)^{k+1}.$$

(b) $E[X] = \sum_{k=0}^\infty \Pr(X>k) = \sum_{k=0}^\infty (1-p)^{k+1} = (1-p)\sum_{k=0}^\infty (1-p)^k$

$$= (1-p)\left[\frac{1}{1-(1-p)}\right] = \boxed{\frac{1-p}{p}}.$$

E2.1.1 $\Pr(X=2 \text{ and } N=3) = \Pr(X=2\,|\,N=3)\Pr(N=3)$
$$= \left[\binom{3}{2}\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^1\right]\left(\tfrac{1}{6}\right) = \boxed{\tfrac{1}{16}}.$$

E2.1.1

$$\Pr(X=5) = \Pr(X=5 \text{ and } N=5) + \Pr(X=5 \text{ and } N=6)$$
$$= \Pr(X=5 \mid N=5)\cdot\Pr(N=5) + \Pr(X=5\mid N=6)\cdot\Pr(N=6)$$
$$= \left(\tfrac{1}{2}\right)^5\cdot\left(\tfrac{1}{6}\right) + \left[\binom{6}{5}\left(\tfrac{1}{2}\right)^5\left(\tfrac{1}{2}\right)^1\right]\left(\tfrac{1}{6}\right)$$
$$= \tfrac{1}{6}\left[\tfrac{1}{32} + \tfrac{3}{32}\right] = \boxed{\tfrac{1}{48}}.$$

$$E[X] = \sum_{n=1}^{6} E[X\mid N=n]\Pr(N=n) \qquad \text{[law of total prob.]}$$
$$= \sum_{n=1}^{6}\left(\tfrac{n}{2}\right)\left(\tfrac{1}{6}\right) = \tfrac{1}{12}[1+2+\cdots+6] = \boxed{\tfrac{7}{4}}$$

E2.1.2 $\Pr(2 \text{ nickels Hs} \mid N=4) = \dfrac{\Pr(2\text{ nickels Hs AND } N=4)}{\Pr(N=4)}$

$$= \frac{\Pr(N=4 \mid 2 \text{ nickels Hs})\Pr(2\text{ nickels Hs})}{\Pr(N=4)}$$
$$= \frac{\binom{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4\cdot\binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2}{\binom{10}{4}\left(\frac{1}{2}\right)^4\left[\frac{1}{2}\right]^6}$$
$$= \frac{\left(\frac{15}{4}\right)(6)\left(\frac{1}{16}\right)}{\left(\frac{210}{64}\right)} = \boxed{\tfrac{3}{7}}$$

E2.1.5 We want $E[X \mid X \text{ is odd}]$. Some preliminary work will be useful. The facts that $e^{\lambda} = \sum_{k=0}^{\infty}\lambda^k/k!$ and $e^{-\lambda} = \sum_{k=0}^{\infty}(-\lambda)^k/k!$ imply

$$\text{(A)}\quad \frac{e^{\lambda}-e^{-\lambda}}{2} = \sum_{\substack{k=0\\ k\text{ odd}}}^{\infty}\lambda^k/k!, \qquad \text{(B)}\quad \frac{e^{\lambda}+e^{-\lambda}}{2} = \sum_{\substack{k=0\\ k\text{ even}}}^{\infty}\lambda^k/k!.$$

Next, let's compute $\Pr(X \text{ is odd})$. We have

$$\Pr(X \text{ is odd}) = \sum_{\substack{k=0\\ k\text{ odd}}}^{\infty}\frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda}\left[\frac{e^{\lambda}-e^{-\lambda}}{2}\right] \qquad \text{[using (A)]}$$
$$= \frac{1-e^{-2\lambda}}{2}.$$

Main computation: $E[X\mid X \text{ is odd}]$

$$= \sum_{\substack{k=0\\ k\text{ odd}}}^{\infty}(k)\Pr(X=k \mid X \text{ is odd})$$
$$= \sum_{j=0}^{\infty}(2j+1)\Pr(X=2j+1 \mid X \text{ is odd})$$
$$= \sum_{j=0}^{\infty}(2j+1)\frac{\Pr(X=2j+1 \text{ AND } X \text{ odd})}{\Pr(X \text{ odd})} = \sum_{j=0}^{\infty}(2j+1)\left[\frac{\lambda^{2j+1}}{(2j+1)!}e^{-\lambda}\right]\cdot\frac{2}{1-e^{-2\lambda}}$$
$$= \frac{2e^{-\lambda}\lambda}{1-e^{-2\lambda}}\sum_{j=0}^{\infty}\frac{\lambda^{2j}}{(2j)!} \underset{[B]}{=} \frac{2\lambda e^{-\lambda}}{1-e^{-2\lambda}}\left(\frac{e^{\lambda}+e^{-\lambda}}{2}\right) = \lambda\left(\frac{e^{\lambda}+e^{-\lambda}}{e^{\lambda}-e^{-\lambda}}\right).$$

P2.1.4 It is shown in the first example of Section 2.1 that X is binomial with parameters M and pq. It follows that $E[X] = Mpq$.

P2.1.5 This is exactly the setting of Problem 2.1.4 with

- X = #heads on tosses of the dime
- $p = \frac{1}{2}$
- N is binomial with parameters $M = 20$ and $q = \frac{1}{2}$.

As above, X is binomial with parameters M and (pq). Thus

$$\Pr(X=0) = \binom{M}{0}(pq)^0(1-pq)^M = (1-pq)^M.$$

P2.1.6 Let's work as follows. A nickel is tossed N times, so X is binomial with parameters N and $p = \frac{1}{2}$.

The number of trials N obeys the distribution

$$p_N(n) = \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^{n-1} = \left(\tfrac{1}{2}\right)^n \quad \text{for } n = 1, 2, \ldots$$

We will use the law of total probability and the summation formulas

$$\text{(A)} \quad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad \text{(B)} \quad \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}, \quad \text{valid for } |x|<1.$$

- $$\Pr(X=0) = \sum_{n=1}^{\infty} \Pr(X=0 \mid N=n)\Pr(N=n) = \sum_{n=1}^{\infty}\left(\tfrac{1}{2}\right)^n\left(\tfrac{1}{2}\right)^n = \sum_{n=1}^{\infty}\left(\tfrac{1}{4}\right)^n \overset{[A]}{=} \frac{\frac{1}{4}}{1-\frac{1}{4}} = \boxed{\tfrac{1}{3}}.$$
- $$\Pr(X=1) = \sum_{n=1}^{\infty} \Pr(X=1 \mid N=n)\Pr(N=n) = \sum_{n=1}^{\infty}\binom{n}{1}\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^{n-1}\cdot\left(\tfrac{1}{2}\right)^n = \sum_{n=1}^{\infty} n\left(\tfrac{1}{4}\right)^n = \left(\tfrac{1}{4}\right)\left[\frac{1}{(1-\frac{1}{4})^2}\right] = \boxed{\tfrac{4}{9}}.$$
- $$E[X] = \sum_{n=1}^{\infty} E[X \mid N=n]\Pr(N=n) = \sum_{n=1}^{\infty}\left(\tfrac{1}{2}n\right)\left(\tfrac{1}{2}\right)^n = \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\sum_{n=1}^{\infty} n\left(\tfrac{1}{2}\right)^{n-1} = \left(\tfrac{1}{4}\right)\left[\frac{1}{(1-\frac{1}{2})^2}\right] = \boxed{1}.$$

P2.1.9 X takes values $0, 1, 2, \ldots$ For k integer, $k \geq 1$, we have

$$\Pr(X=k) = \sum_{n=k-1}^{\infty} \Pr(X=k \mid N=n)\Pr(N=n) = \sum_{n=k-1}^{\infty}\left(\frac{1}{n+2}\right)\frac{\lambda^n}{n!}e^{-\lambda}$$
$$= e^{-1}\sum_{n=k-1}^{\infty}\left(\frac{1}{n+2}\right)\left(\frac{1}{n!}\right) = e^{-1}\left(\frac{1}{k!}\right). \quad \text{[WHY?]}$$

For $k=0$, $\Pr(X=0) = \Pr(X=1)$. Thus the marginal distribution for X is Poisson w/ parameter 1 !