Math 4320 Fall 2016 | Assignment 4 Solutions

E2.3.1 Z is binomial w/ probability of success $\frac{1}{2}$ and number of trials N. N has probability mass function $p_N(i) = \frac{1}{6}$ for i in $\{1, \dots, 6\}$. Let A_i be the event that heads appears on the i^{th} flip and let $\mathbb{1}_{A_i} = \begin{cases} 1, & \text{if the } i^{\text{th}} \text{ flip is heads;} \\ 0, & \text{otherwise.} \end{cases}$

Then $Z = \mathbb{1}_{A_1} + \mathbb{1}_{A_2} + \dots + \mathbb{1}_{A_N}$ is a random sum.

Each of the $\mathbb{1}_{A_i}$ is Bernoulli with mean p and variance $p(1-p)$. For N, $E[N] = \sum_{i=1}^{6} i\left(\frac{1}{6}\right) = \frac{21}{6} = \frac{7}{2}$. For the variance, we need $E[N^2] = \sum_{i=1}^{6} i^2\left(\frac{1}{6}\right) = \frac{91}{6}$. Thus $\text{Var}[N] = \frac{91}{6} - \left[\frac{7}{2}\right]^2 = \frac{35}{12}$.

Using formulas (2.30),

$$E[Z] = E[\mathbb{1}_{A_i}]E[N] = p \cdot E[N] = \left(\tfrac{1}{2}\right)\left(\tfrac{7}{2}\right) = \tfrac{7}{4}.$$

$$\begin{aligned}\text{Var}[Z] &= E[N]\,\text{Var}[\mathbb{1}_{A_i}] + (E[\mathbb{1}_{A_i}])^2\,\text{Var}[N] \\ &= \left(\tfrac{7}{2}\right)\left(\tfrac{1}{4}\right) + \left(\tfrac{1}{4}\right)\left(\tfrac{35}{12}\right) = \tfrac{77}{48}.\end{aligned}$$

E2.3.3 Let ξ_i = # of electrons released by the i^{th} daughter electron. Then $Z = \xi_1 + \dots + \xi_N$, so (2.30) gives

$$E[Z] = (\mu)(\mu) = \mu^2.$$

$$\text{Var}[Z] = (\mu)\sigma^2 + \mu^2\sigma^2 = \mu\sigma^2(1+\mu).$$

E2.3.5 Let X = # of individuals injured in a week, N = # of accidents per week, and ξ_i = # individuals injured in the i^{th} accident. Then $X = \xi_1 + \dots + \xi_N$. Using (2.30),

$$E[X] = E[\xi_i]E[N] = (3)(2) = 6.$$

$$\begin{aligned}\text{Var}[X] &= E[N]\,\text{Var}[\xi_i] + (E[\xi_i])^2\,\text{Var}[N] = \\ &= (2)(4) + (9)(2) = 26.\end{aligned}$$

P2.3.2 Let $A_1, \dots, A_n, \dots$ be independent events, each w/ probability p. Then $Z = \mathbb{1}_{A_1} + \dots + \mathbb{1}_{A_N}$ is a random sum.

We have already used conditioning to show that the marginal distribution of Z is binomial. See page 48.

E3.1.1 $\Pr\{X_0 = 0, X_1 = 1, X_2 = 2\} = p_0 P_{01} P_{12} = (0.3)(0.2)(0) = 0.$

E3.1.4 $\Pr\{X_1 = 1, X_2 = 1 \mid X_0 = 0\} = P_{01} P_{11} = (0.1)(0.2) = .02.$

$\Pr\{X_2 = 1, X_3 = 1 \mid X_1 = 0\} = P_{01} P_{11} = .02.$

E3.1.5 $\Pr\{X_0=1, X_1=1, X_2=0\} = (0.5)P_{11}P_{10} = (0.5)(0.1)(0.5)$
$= .025.$

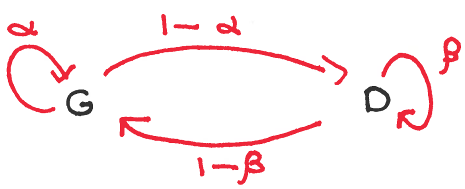
$\Pr\{X_1=1, X_2=1, X_3=0\}$

$= p_0 P_{01} P_{11} P_{10} + p_1 P_{11} P_{11} P_{10}$

$= (\tfrac{1}{2})(0.2)(0.1)(0.5) + (\tfrac{1}{2})(0.1)(0.1)(0.5)$

$= \tfrac{1}{2}(.01 + .005) = .0075.$

P3.1.3 The Markov chain is described graphically by



If the first item is good (we are currently in state G), the prob. that the first defective item to appear is the fifth item:

$\alpha^3(1-\alpha).$

P3.1.4

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.1 & 0.3 & 0.2 & 0.4 \\ 0 & 0.4 & 0.2 & 0.4 \\ 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

E3.2.4 $\Pr(X_2=2 \mid X_0=1) = P_{10}P_{02} + P_{11}P_{12} + P_{12}P_{22}$

$= (0.3)(0.1) + (0.3)(0.4) + (0.4)(0.5)$

$= 0.35.$

E3.2.6 We have $P^2 = \begin{pmatrix} .44 & .18 & .38 \\ .4 & .19 & .41 \\ .4 & .18 & .42 \end{pmatrix}$, $P^3 = \begin{pmatrix} .412 & .182 & .406 \\ .42 & .181 & .399 \\ .42 & .182 & .398 \end{pmatrix}$.

* $\Pr\{X_2=0\} = (0.5)(.44) + (0.5)(.4) = .42.$

* $\Pr\{X_3=0\} = (0.5)(.412) + (0.5)(.42) = .416.$

P3.2.5 We have

$$P^3 = \begin{pmatrix} .457 & .23 & .313 \\ .345 & .227 & .428 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\Pr(X_3=0 \mid X_0=0, T>3) = \Pr(X_3=0 \mid X_0=0, X_3=0 \text{ or } 1)$

$= \dfrac{(.457)}{.457+.23} = \dfrac{457}{687} \approx .6652.$