E3.3.2 Let (X_n) track the number of balls in A. The state space is 0 (all balls in urn B), $1, 2, \ldots, N-1$, N (all balls in urn A).

$$
P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 0 & 1 & 2 & 3 & \cdots\ N-2 & N-1 & N \end{array} \\
\begin{pmatrix}
q & p & & & & & \\
l_1 & s_1 & r_1 & & & & \\
 & l_2 & s_2 & r_2 & & & \\
 & & & \ddots & & & \\
 & & & & l_{N-1} & s_{N-1} & r_{N-1} \\
 & & & & & q & p
\end{pmatrix}
\end{array}
$$

For $1 \le k \le N-1$, we have

$$l_k = \left(\frac{k}{N}\right) q$$

$$r_k = \left(\frac{N-k}{N}\right) p$$

$$s_k = \left(\frac{k}{N}\right) p + \left(\frac{N-k}{N}\right) q .$$

E3.3.5 Let X_n = # red balls in the urn after n draws. $[X_0 = 1]$

$$
P = \begin{array}{c|ccc}
 & 0 & 1 & 2 \\ \hline
0 & 0 & 1 & 0 \\
1 & \frac{1}{2} & 0 & \frac{1}{2} \\
2 & 0 & 1 & 0
\end{array}
$$

P3.3.1 Let X_n = # red tags in the urn after the n^{th} draw. $[X_0 = 3]$.

$$
P = \begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\ \hline
0 & 1 & 0 & 0 & 0 \\
1 & \frac{1}{15} & \frac{14}{15} & 0 & 0 \\
2 & 0 & \frac{\binom{2}{1}\binom{2}{1}}{15} = \frac{4}{15} & \frac{11}{15} & 0 \\
3 & 0 & 0 & \frac{\binom{3}{1}\binom{3}{1}}{\binom{6}{2}} = \frac{3}{5} & \frac{2}{5}
\end{array}
$$

P3.3.5

$$
P = \begin{array}{c|cccc}
 & 0 & H & HH & HHT \\ \hline
0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
H & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
HH & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
HHT & 0 & 0 & 0 & 1
\end{array}
$$

| P3.3.6 |

$$P_{(a,b),(c,d)} = \begin{cases} p, & \text{if } c = a+1,\ b = d, \text{ and } \max\{a,b\} \le 3; \\ 1-p, & \text{if } a = c,\ d = b+1 \text{ and } \max\{a,b\} \le 3; \\ 1, & \text{if } a = c,\ b = d, \text{ and } (a=4 \text{ or } b=4); \\ 0, & \text{otherwise.} \end{cases}$$

| P3.3.8 | Suppose the Markov chain tracks the number of balls in urn A.

$$P = \begin{array}{c|c} & \begin{array}{ccccccc} 0 & 1 & 2 & 3 \cdots & N-2 & N-1 & N \end{array} \\ \hline \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array} & \begin{pmatrix} 1 & & & & & & \\ \ell_1 & s_1 & r_1 & & & & \\ & \ell_2 & s_2 & r_2 & & & \\ & & & \ddots & & & \\ & & & & \ell_{N-1} & s_{N-1} & r_{N-1} \\ & & & & & & 1 \end{pmatrix} \end{array}$$

For $1 \le k \le N-1$, we have

$$\ell_k = \left(\frac{N-k}{N}\right)p$$
$$s_k = \left(\frac{k}{N}\right)p + \left(\frac{N-k}{N}\right)q$$
$$r_k = \left(\frac{k}{N}\right)q.$$

| E3.4.1 | Let T denote the absorbing time and let $v_i = E[T \mid X_0 =$ i [redacted] $]$.
The first-step system of equations for v_0, v_1, v_2 is

$$\begin{cases} v_0 = 1 + 0.4v_0 + 0.3v_1 + 0.2v_2 \\ v_1 = 1 + 0.7v_1 + 0.2v_2 \\ v_2 = 1 + 0.9v_2. \end{cases}$$

The solution is $v_0 = v_1 = v_2 = 10$. In particular, $v_0 = 10$.

| E3.4.3 | (a) Let T denote absorption time. Define $u_1 = \Pr(X_T = 0 \mid X_0 = 1)$ and $u_2 = \Pr(X_T = 0 \mid X_0 = 2)$. First step analysis gives

$$\begin{cases} u_1 = 0.1 + 0.6u_1 + 0.1u_2 \\ u_2 = 0.2 + 0.3u_1 + 0.4u_2 \end{cases}$$

The solution is $u_1 = \frac{8}{21}$, $u_2 = \frac{11}{21}$. In particular, $u_1 = \frac{8}{21}$.

E3.4.3 continued We find $E[T \mid X_0 = 1]$. Define $v_i := E[T \mid X_0 = i]$.

First-step analysis gives
$$\begin{cases} v_1 = 1 + 0.6 v_1 + 0.1 v_2 \\ v_2 = 1 + 0.3 v_1 + 0.4 v_2 . \end{cases}$$

The solution is $v_1 = v_2 = 10/3$. In particular, $v_1 = 10/3$.

E3.4.7 First define α_1 = mean time in state 1 prior to absorption $\mid X_0 = 1$,

α_2 = mean time in state 1 prior to absorption $\mid X_0 = 2$.

First-step analysis gives
$$\begin{cases} \alpha_1 = 1 + 0.2\alpha_1 + 0.5\alpha_2 \\ \alpha_2 = 0.2\alpha_1 + 0.6\alpha_2 \end{cases}$$

Solution: $\alpha_1 = \frac{20}{11}$, $\alpha_2 = 10/11$. In particular, $\boxed{\alpha_1 = 20/11.}$

Now define $\beta_i :=$ mean time in state 2 prior to absorption $\mid X_0 = i$.

First-step analysis gives:
$$\begin{cases} \beta_1 = 0.2\beta_1 + 0.5\beta_2 \\ \beta_2 = 1 + 0.2\beta_1 + 0.6\beta_2 \end{cases}$$

Solution: $\boxed{\beta_1 = 25/11}$, $\beta_2 = 40/11$. In particular, $\beta_1 = 25/11$.

Finally, let $v_i := E[T \mid X_0 = i]$. First-step [redacted] analysis gives
$$\begin{cases} v_1 = 1 + 0.2 v_1 + 0.5 v_2 \\ v_2 = 1 + 0.2 v_1 + 0.6 v_2 \end{cases} \qquad \text{Solution: } v_1 = 45/11, \quad v_2 = 50/11.$$

Notice that $v_1 = \alpha_1 + \beta_1$.

P3.4.1 Let's set up a Markov chain to analyze each situation.

For the HHT case, we set up such a chain in P3.3.5:

$$P = \begin{array}{c} \\ 0 \\ H \\ HH \\ HHT \end{array} \begin{array}{c} \begin{array}{cccc} 0 & H & HH & HHT \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Let T denote absorption time. Define $v_i := E[T \mid X_0 = i]$.

First-step equations:
$$\begin{cases} v_0 = 1 + \frac{1}{2} v_0 + \frac{1}{2} v_H \\ v_H = 1 + \frac{1}{2} v_0 + \frac{1}{2} v_{HH} \\ v_{HH} = 1 + \frac{1}{2} v_{HH} \end{cases}$$

Solution : $v_0 = 8$, $v_H = 6$, $v_{HH} = 2$. In particular, $\boxed{v_0 = 8!}$

For the HTH case, analyze the following Markov chain :

$$\underline{P} = \begin{array}{c} \\ O \\ H \\ HT \\ HTH \end{array} \begin{array}{c} \begin{array}{cccc} O & H & HT & HTH \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Let $w_i = E[T \mid X_0 = i]$. First-step system:

$$\begin{cases} w_0 = 1 + \frac{1}{2} w_0 + \frac{1}{2} w_H \\ w_H = 1 + \frac{1}{2} w_H + \frac{1}{2} w_{HT} \\ w_{HT} = 1 + \frac{1}{2} w_0 + 0 \end{cases}$$

Solution : $w_0 = 10$
$w_H = 8$
$w_{HT} = 6$.

In particular, $\boxed{w_0 = 10}$.

Conclusion : It takes 10 flips (on avg.) to see HTH but only 8 to see HHT. WHY?

$\boxed{\text{P3.4.4}}$ Define $y_i := \Pr(\text{process never visits state } 2 \mid X_0 = i)$.

First-step analysis gives :

$$\begin{cases} y_1 = (0.1)(1) + 0.2\, y_1 + (0.2) y_3 \\ y_3 = (0.2)(1) + 0.2\, y_1 + (0.3) y_3 \end{cases}$$

Solution : $y_1 = 11/52$, $y_2 = 9/26$. In particular, $\boxed{y_1 = 11/52.}$

$\boxed{\text{P3.4.5}}$ Define $u_i = \Pr(\text{food before shock} \mid X_0 = i)$.

First-step system :

$$\begin{cases} u_1 = \frac{1}{2} u_2 + \frac{1}{2} u_4 \\ u_2 = \frac{1}{3} + \frac{1}{3} u_1 + \frac{1}{3} u_5 \\ u_4 = \frac{1}{3} u_1 + \frac{1}{3} u_5 \\ u_5 = \frac{1}{3} u_2 + \frac{1}{3} u_4 + \frac{1}{3} u_6 \\ u_6 = \frac{1}{2} + \frac{1}{2} u_5 \end{cases}.$$

Solution:
$u_1 = 7/12$
$u_2 = 3/4$
$u_4 = 5/12$
$u_5 = 2/3$
$u_6 = 5/6$.

In particular, $\boxed{u_4 = 5/12.}$

$\boxed{\text{P3.4.6}}$ Let $v_i = E[T \mid X_0 = i]$. First-step analysis gives:

$$\begin{cases} v_0 = 1 + q v_0 + p v_1 \\ v_1 = 1 + q v_0 + p v_2 \\ v_2 = 1 + q v_0 + p v_3 \\ v_3 = 1 + q v_0 + p v_4 \\ v_4 = 0 \end{cases}$$

Solving from the bottom up, we have

$$\begin{aligned} v_3 &= 1 + q v_0 ; \\ v_2 &= 1 + q v_0 + p[1 + q v_0] \\ &= [1 + q v_0](1+p). \end{aligned}$$

Continuing,

$$\begin{aligned} v_1 &= [1+q v_0] + p[[1+q v_0](1+p)] \\ &= [1+q v_0](1+p+p^2). \end{aligned}$$

We arrive at

$$v_0 = [1+q v_0](1+p+p^2+p^3).$$

Thus

$$v_0 = \frac{1+p+p^2+p^3}{1-q[1+p+p^2+p^3]}.$$