P3.4.9 Let's track the # of red balls remaining. The transition matrix:

$$
P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{pmatrix}
1 & & & & & \\
\frac{1}{8} & \frac{7}{8} & & & & \\
 & \frac{1}{4} & \frac{3}{4} & & & \\
 & & \frac{3}{8} & \frac{5}{8} & & \\
 & & & \frac{1}{2} & \frac{1}{2} & \\
 & & & & \frac{5}{8} & \frac{3}{8}
\end{pmatrix}
\end{array}.
$$

Let T be the absorbing time and define $v_i = E[T \mid X_0 = i]$.
First step analysis gives:

$$
\begin{cases}
v_1 = 1 + \frac{1}{8}\cdot 0 + \frac{7}{8}v_1 \\
v_2 = 1 + \frac{1}{4}v_1 + \frac{3}{4}v_2 \\
v_3 = 1 + \frac{3}{8}v_2 + \frac{5}{8}v_3 \\
v_4 = 1 + \frac{1}{2}v_3 + \frac{1}{2}v_4 \\
v_5 = 1 + \frac{5}{8}v_4 + \frac{3}{8}v_5
\end{cases}
$$

Solution: $v_1 = 8$, $v_2 = 12$, $v_3 = {}^{44}\!/_3$, $v_4 = {}^{50}\!/_3$, $v_5 = {}^{274}\!/_{15}$.
Thus $v_5 = {}^{274}\!/_{15}$ is the answer to the original question.

P3.4.12 Let T denote absorbing time. We want $\Pr(X_{T-1} = 1 \mid X_0 = 0)$.
Define $z_i = \Pr(X_{T-1} = 1 \mid X_0 = i)$. We want z_0. First step analysis:

$$
\begin{cases}
z_0 = 0.3z_0 + 0.2z_1 \\
z_1 = 0.4 + 0.5z_0 + 0.1z_1
\end{cases}
$$

Solution: $z_0 = {}^{8}\!/_{53}$, $z_1 = {}^{28}\!/_{53}$. In particular, $z_0 = {}^{8}\!/_{53}$.

E3.5.4

$$
P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & & \\
\frac{1}{2} & & \frac{1}{2} & \\
\frac{1}{2} & & & \frac{1}{2} \\
 & & & 1
\end{pmatrix}
\end{array}
$$

E3.5.6 Let $X_n = \max\{\xi_1, \dots, \xi_n\}$. Time of sale is $T = \min\{n \geq 1 : X_n \geq M\}$.
First step analysis for $\mu = E[T]$ gives $\mu = 1 + \mu \Pr(\xi_1 < M)$, or

$$
\mu = \frac{1}{\Pr(\xi_1 \geq M)} = \frac{1}{\sum_{k=M}^{\infty}(0.01)(0.99)^k} = \frac{100}{(0.99)^M\left[\frac{1}{1-0.99}\right]} = (0.99)^{-M}.
$$

In particular, $M = 100$ gives $\mu \approx 2.73$.

P3.5.1 Define $\nu_i = E[T \mid X_0 = i]$ for $i = 0,1,2$. Here T is absorption time.

First step analysis:
$$\begin{cases} \nu_0 = 1 + a_0\nu_0 + a_1\nu_1 + a_2\nu_2 \\ \nu_1 = 1 \qquad\quad + (a_0+a_1)\nu_1 + a_2\nu_2 \\ \nu_2 = 1 \qquad\qquad\qquad + (a_0+a_1+a_2)\nu_2 . \end{cases}$$

Solution:
$$\nu_2 = \frac{1}{1-(a_0+a_1+a_2)} = \frac{1}{a_3}.$$
$$\nu_1 = \frac{1 + a_2/a_3}{1-(a_0+a_1)} = \frac{\frac{a_3+a_2}{a_3}}{a_2+a_3} = \frac{1}{a_3}.$$
$$\nu_0 = \frac{1 + a_1/a_3 + a_2/a_3}{1-a_0} = \frac{(a_1+a_2+a_3)/a_3}{a_1+a_2+a_3} = \frac{1}{a_3}.$$

P3.5.3
$$\underline{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \alpha & 1-\alpha & 0 \\ 0 & \alpha & 1-\alpha \\ 1-\alpha & 0 & \alpha \end{pmatrix} \end{array}.$$

E3.8.2 $E[\xi] = 0\cdot a + 1\cdot b + 2\cdot c = b + 2c$

$$\begin{aligned} Var[\xi] &= E[\xi^2] - (E[\xi])^2 = [0\cdot a + 1\cdot b + 4\cdot c] - [b+2c]^2 \\ &= b + 4c - (b^2 + 4bc + 4c^2) \\ &= \text{(no further simplifications)} \end{aligned}$$

E3.8.3 Let ξ = # of offspring. Then $Pr(\xi = 0) = \frac{1}{2}$ and $Pr(\xi = 2) = \frac{1}{2}$.

First step analysis of the branching process gives
$$u_n = \tfrac{1}{2} + \tfrac{1}{2}(u_{n-1})^2 \quad \text{for } n \geq 1.$$
We therefore have $u_1 = \frac{1}{2}$, $u_2 = 5/8$, $u_3 = \frac{89}{128}$, $u_4 = \frac{24305}{32768}$,
$u_5 = 1664474849/2147483648$.

P3.8.1

parent

L offspring stay — all live w/ prob. β, all die w/ prob. $1-\beta$

$M-L$ disperse ($= N$) — each lives w/ prob. $p\beta = \alpha$, independently of the others

- $E[\xi] = \sum_{i=1}^{N} E[\nu_i] + (M-N)E[\theta]$ $= \alpha N + \beta(M-N)$.

  highest when $N=0$

- $Pr[\xi = 0] = (1-\beta)(1-\alpha)^N$

  smallest when $N = M$.

P3.8.2
$$\begin{aligned} E[Z] &= E\Big[\sum_{n=0}^{\infty} X_n\Big] \\ &= \sum_{n=0}^{\infty} E[X_n] \\ &= \sum_{n=0}^{\infty} \mu^n = \frac{1}{1-\mu}. \end{aligned}$$