E3.9.1 The generating function of the offspring dist. is $\phi(s) = e^{-\lambda(1-s)}$. (See pg. 154) So we have

$$u_0 = 0$$
$$u_1 = \frac{(1.1)^0}{0!} e^{-1.1} = e^{-\lambda} \approx 0.333$$
$$u_2 = \phi(u_1) = e^{-\lambda(1-e^{-\lambda})} \approx 0.480$$
$$u_3 = \phi(u_2) = e^{-\lambda(1-u_2)} \approx 0.564$$
$$u_4 = \phi(u_3) = e^{-\lambda(1-u_3)} \approx 0.619$$
$$u_5 = \phi(u_4) = e^{-\lambda(1-u_4)} \approx 0.658$$

- u_∞ is the smallest positive solution to $\phi(s) = s$, or $e^{-\lambda(1-s)} = s$. Numerically, $u_\infty \approx 0.824$.

P3.9.1 Let ξ denote the number of boys produced by a given husband. We have the following mass function:

ξ	Pr
0	$1/4 + (3/4)\cdot(1/8) = 11/32$
1	$(3/4)\binom{3}{1}(1/2)^1(1/2)^2 = 9/32$
2	$(3/4)\binom{3}{2}(\frac{1}{2})^2(\frac{1}{2})^1 = 9/32$
3	$(3/4)(1/8) = 3/32$

Consequently, the generating function associated with ξ is $\phi(s) = 11/32 + 9/32\,s + 9/32\,s^2 + \frac{3}{32}s^3$. To find u_∞, solve $\phi(s) = s$:

$$11 + 9s + 9s^2 + 3s^3 = 32s$$
$$\Rightarrow \quad 3s^3 + 9s^2 - 23s + 11 = 0$$

Solutions: $1,\ -2 \pm \sqrt{23/3}$.

So $u_\infty = -2 + \sqrt{23/3} \approx 0.769$, the smallest positive solution.

E4.1.2 Solve $\begin{cases} (\pi_0, \pi_1, \pi_2) P = (\pi_0, \pi_1, \pi_2) \\ \pi_0 + \pi_1 + \pi_2 = 1. \end{cases}$

Solution: $\pi_0 = 31/66$, $\pi_1 = 8/33$, $\pi_2 = 19/66$.

E4.1.8 Solve $\begin{cases} (\pi_L, \pi_M, \pi_U) P = (\pi_L, \pi_M, \pi_U) \\ \pi_L + \pi_M + \pi_U = 1. \end{cases}$

Limiting distribution: $(\pi_L, \pi_M, \pi_U) = (6/17, 7/17, 4/17)$.

Thus the fraction of families in the upper class in the long run is $\pi_U = 4/17$.

P4.1.1 Let X_n denote the number of balls in urn A at time n. The transition matrix for (X_n) is

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1/2 & 1/2 & & & & \\ 1/2 & & 1/2 & & & \\ & 1/2 & & 1/2 & & \\ & & 1/2 & & 1/2 & \\ & & & 1/2 & & 1/2 \\ & & & & 1/2 & 1/2 \end{pmatrix} \end{array}.$$

Solving $\begin{cases} \pi P = \pi \\ \sum_{i=0}^{5} \pi_i = 1 \end{cases}$ for the limiting dist. gives

$$(\pi_0, \pi_1, \pi_2, \pi_3, \pi_4) = \left(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}\right)$$
and $\pi_5 = \frac{1}{6}$.

In particular, urn A is empty $\pi_0 = \frac{1}{6}$ of the time in the long run.

P4.1.2 With (X_n) as in P4.1.1, the transition matrix is now

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} & 1 & & & & \\ 1/5 & & 4/5 & & & \\ & 2/5 & & 3/5 & & \\ & & 3/5 & & 2/5 & \\ & & & 4/5 & & 1/5 \\ & & & & 1 & \end{pmatrix} \end{array}.$$

The limiting distribution is $(\pi_0, \ldots, \pi_5) = \left(\frac{1}{32}, \frac{5}{32}, \frac{5}{16}, \frac{5}{16}, \frac{5}{32}, \frac{1}{32}\right)$.

Urn A is empty $\pi_0 = \frac{1}{32}$ of the time in the long run.

P4.1.4 fraction of transitions from k to m in the long run: $\pi_k P_{km}$

P4.1.5 States: A, B, C, D

Transition matrix:

$$P = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} & 1/2 & & 1/2 \\ 1/3 & & 1/3 & 1/3 \\ & 1 & & \\ 1/2 & 1/2 & & \end{pmatrix} \end{array}.$$

The limiting distribution is $(\pi_A, \pi_B, \pi_C, \pi_D) = \left(\frac{1}{4}, \frac{3}{8}, \frac{1}{8}, \frac{1}{4}\right)$.

In particular, the long run prob. of finding the train at station D $= \pi_D = \frac{1}{4}$.

P4.1.6 (a) π_j (b) $\pi_k P_{kj}$ (c) $\pi_k P_{kj}$

P4.1.13

$$\lim_{n\to\infty} \Pr(X_{n-1}=2 \mid X_n=1)$$

$$= \lim_{n\to\infty} \frac{\Pr(X_{n-1}=2 \text{ AND } X_n=1)}{\Pr(X_n=1)}$$

$$= \lim_{n\to\infty} \frac{\Pr(X_n=1 \mid X_{n-1}=2)\Pr(X_{n-1}=2)}{\Pr(X_n=1)}$$

$$= \frac{P_{21}\pi_2}{\pi_1},$$ where (π_0, π_1, π_2) denotes the limiting distribution.

In this case, $(\pi_0,\pi_1,\pi_2) = \left(\frac{11}{24}, \frac{7}{24}, \frac{1}{4}\right)$, so

$$\frac{P_{21}\pi_2}{\pi_1} = \frac{\left(\frac{1}{5}\right)\left(\frac{1}{4}\right)}{\frac{7}{24}} = \frac{6}{35}.$$

P4.2.4 (a)

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \\ \end{array} \begin{array}{c} \left(\begin{array}{cccc} 0.1 & 0.3 & 0.2 & 0.4 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{array}\right) \\ \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \end{array}.$$

(b) P is regular, since P^4 has all positive entries.
The limiting dist. is $(\pi_0, \pi_1, \pi_2, \pi_3) = \left(\frac{10}{29}, \frac{9}{29}, \frac{6}{29}, \frac{4}{29}\right)$.
In particular, we want $\pi_0 = \frac{10}{29}$.

(c) $E[\xi] = 0.1 + 0.6 + 0.6 + 1.6 = \frac{29}{10} = \frac{1}{\pi_0}$.