P4.3.1 (a) $f_{00}^{(n)} = a(1-b)^{n-2}b$ for $n \geq 2$;
$f_{00}^{(1)} = 1-a$.

(b) We check (4.16) for the general case $0 < a, b < 1$ and $a+b \neq 1$.
(4.16) holds by inspection for $n=1$. For $n \geq 2$:

$$\sum_{k=1}^{n} f_{00}^{(k)} P_{00}^{(n-k)} = \sum_{k=1}^{n} f_{00}^{(k)} \left[\left(\frac{1}{a+b}\right)\left(b + [1-a-b]^{n-k} a\right)\right] \qquad \text{(Eq. 3.31)}$$

$$= (1-a)\left[\left(\frac{1}{a+b}\right)\left(b + [1-a-b]^{n-1} a\right)\right] + \sum_{k=2}^{n} \left(a(1-b)^{k-2} b\right)\left[\left(\frac{1}{a+b}\right)\left(b + [1-a-b]^{n-k} a\right)\right]$$

$$= \left(\frac{1}{a+b}\right)\left[(1-a)b + (1-a)a[1-a-b]^{n-1} + ab^2\left(\frac{1-(1-b)^{n-1}}{1-(1-b)}\right) + a^2 b\left(\frac{[1-a-b]^n}{(1-b)^2}\right)\sum_{k=2}^{n}\left[\frac{1-b}{1-a-b}\right]^k\right]$$

$$= \left(\frac{1}{a+b}\right)\left[(1-a)b + (1-a)a[1-a-b]^{n-1} + ab^2\left(\frac{1-(1-b)^{n-1}}{b}\right) + a^2 b[1-a-b]^{n-2}\left[\frac{1-\left(\frac{1-b}{1-a-b}\right)^{n-1}}{1-\left(\frac{1-b}{1-a-b}\right)}\right]\right]$$

$$= \left(\frac{1}{a+b}\right)\left[b + a[1-a-b]^n\right] = P_{00}^{(n)} \qquad \text{(Eq. 3.31)}.$$

E5.1.1 (a) Let $N([a,b])$ denote the number of defects in $[a,b]$.
Then $\Pr(N([0,1]) = 0) = \frac{(\lambda \cdot 1)^0}{0!} e^{-\lambda} = e^{-\lambda}$.

(b) $\Pr(N([1,2]) = 0 \mid N([0,1]) = 1) = e^{-\lambda}$.

E5.1.4 (a) $\Pr(X(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$

(b) $\Pr(X(t) = n+k \mid X(s) = n)$

$$= \frac{[\lambda(t-s)]^k}{k!} e^{-\lambda(t-s)}$$

$$E[X(t)X(s)] = E[(X(s) + [X(t) - X(s)]) \cdot X(s)]$$
$$= E[X^2(s)] + E[(X(t) - X(s))X(s)]$$
$$= \lambda s + (\lambda s)^2 + (\lambda(t-s))(\lambda s)$$

E5.1.5

$$\Pr(X=k) = \int_0^\infty \Pr(X=k \mid \lambda = x)\, \theta e^{-\theta x}\, dx$$
$$= \int_0^\infty \left[x^k/k!\, e^{-x} \right] \theta e^{-\theta x}\, dx$$
$$= \theta/k! \int_0^\infty x^k e^{-(1+\theta)x}\, dx$$
$$= \frac{\theta}{(1+\theta)^{k+1} \cdot k!} \int_0^\infty u^k e^{-u}\, du \qquad [u = (1+\theta)x]$$
$$= \frac{\theta}{(1+\theta)^{k+1} \cdot k!} \Gamma(k+1)$$
$$= \frac{\theta}{(1+\theta)^{k+1}} . \qquad [\Gamma(k+1) = k!]$$

E5.1.7 (a) $\Pr(X(1)=2) = [\lambda \cdot 1]^2/2!\, e^{-\lambda}$

(b) $\Pr(X(1)=2 \text{ and } X(3)=6)$

$$= \left((\lambda \cdot 1)^2/2!\, e^{-\lambda} \right) \left((2\lambda)^4/4!\, e^{-2\lambda} \right)$$

(c) $\Pr(X(1)=2 \mid X(3)=6)$

$$= \Pr(X(1)=2 \text{ and } X(3)=6) / \Pr(X(3)=6)$$
$$= \frac{\left((\lambda \cdot 1)^2/2!\, e^{-\lambda} \right) \left((2\lambda)^4/4!\, e^{-2\lambda} \right)}{\left((3\lambda)^6/6!\, e^{-3\lambda} \right)}$$

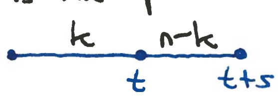
(d) $\Pr(X(3)=6 \mid X(1)=2) = (2\lambda)^4/4!\, e^{-2\lambda}$.

P5.1.3 $g(s) = \sum_{k=0}^\infty \left(\mu^k/k!\, e^{-\mu} \right) s^k = e^{-\mu} \sum_{k=0}^\infty \frac{(\mu s)^k}{k!} = e^{-\mu} \cdot e^{\mu s} = e^{-\mu(1-s)}$.

P5.1.4 $g_{X+Y}(s) = g_X(s) g_Y(s) = e^{-\alpha(1-s)} \cdot e^{-\beta(1-s)}$
$$= e^{-(\alpha+\beta)(1-s)} .$$

This works because the generating function of a sum of independent RVs is the product of the individual generating functions.

P5.1.6



Given that $X(t+s) = n$, $X(t)$ may take any value in $\{0, \dots, n\}$.

For k in $\{0, \dots, n\} \Rightarrow \Pr(X(t)=k \mid X(t+s)=n) = \dfrac{\Pr(X(t)=k \text{ AND } X(t+s)=n)}{\Pr(X(t+s)=n)}$

$$= \frac{\left[(\lambda t)^k/k!\, e^{-\lambda t} \right] \left[\frac{(\lambda s)^{n-k}}{(n-k)!} e^{-\lambda s} \right]}{[\lambda(t+s)]^n/n!\, e^{-\lambda(t+s)}} = \binom{n}{k} \frac{t^k s^{n-k}}{(t+s)^n} .$$

P5.1.7

$$
\begin{aligned}
&\Pr(\text{survival at time } t) \\
&= \sum_{k=0}^{\infty} \Pr(\text{survival at time } t \mid k \text{ shocks}) \Pr(k \text{ shocks by time } t) \\
&= \sum_{k=0}^{\infty} \alpha^k \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\
&= e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\alpha \lambda t)^k}{k!} \\
&= e^{-\lambda t} e^{\alpha \lambda t} \\
&= e^{-\lambda t (1-\alpha)} .
\end{aligned}
$$

P5.2.4 Each of the N points lies in $[0,1)$ w/ probability $\frac{1}{N}$. The number of points in $[0,1)$ is binomially distributed w/ N trials and probability of "success" $\frac{1}{N}$. According to the law of rare events, the limiting distribution as $N \to \infty$ is Poisson with mean 1.