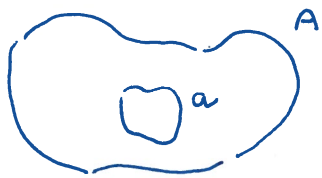
P5.2.7



The number of bacteria in the subregion is binomially distributed w/ # of trials N and probability of success a/A.

So PR (subregion contains k bacteria) $= \binom{N}{k}\left(\frac{a}{A}\right)^k\left(1-\frac{a}{A}\right)^{N-k}$.

If we take the $N \to \infty$, $a \to 0$ limit and require that $(a/A)N \to c$ for some $0<c<\infty$, we obtain convergence to the Poisson dist.:

$$\Pr(\text{subregion contains } k \text{ bacteria}) \to \frac{c^k}{k!}e^{-c}.$$

E5.3.1 $\Pr(W_1 > 3\text{ min}) = \int_3^\infty \lambda e^{-\lambda t}\,dt = -e^{-\lambda t}\Big|_3^\infty = e^{-3\lambda}$.

E5.3.3 $\Pr(0 \le W_1 \le \frac{1}{4} \mid X(1)=1)$

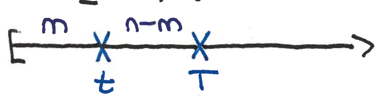
$$= \Pr\left(0 \le W_1 \le \tfrac14 \text{ AND } X(1)=1\right) / \Pr(X(1)=1)$$

$$= \Pr\left(X(\tfrac14)=1 \text{ AND } X(1)-X(\tfrac14)=0\right) / \Pr(X(1)=1)$$

$$= \left(\left(\tfrac14\lambda\right)e^{-\frac14\lambda}\right)\left(\frac{(\frac34\lambda)^0}{0!}e^{-\frac34\lambda}\right) \Big/ \left(\frac{\lambda^1}{1!}e^{-\lambda}\right)$$

$$= \frac14.$$

E5.3.5



$$\Pr(X(t)=m \mid X(T)=n) = \frac{\Pr(X(t)=m \text{ AND } X(T)-X(t)=n-m)}{\Pr(X(T)=n)}$$

$$= \left[\frac{(\theta t)^m}{m!}e^{-\theta t}\right]\left[\frac{[\theta(T-t)]^{n-m}}{(n-m)!}e^{-\theta(T-t)}\right] \Big/ \left[\frac{(\theta T)^n}{n!}e^{-\theta T}\right]$$

$$= \binom{n}{m}\frac{t^m(T-t)^{n-m}}{T^n}.$$ (One can write this as a binomial dist.)

E5.3.7 $E[W_5 \mid X(t)=3] = t + 2/\lambda$.

P5.3.5 For integer $k \ge 0$, we compute $\Pr(X(T)=k)$ using the law of total probability:

$$\Pr(X(T)=k) = \int_0^\infty \Pr(X(s)=k)\,\theta e^{-\theta s}\,ds$$

$$= \int_0^\infty \left[\frac{(\lambda s)^k}{k!}e^{-\lambda s}\right]\theta e^{-\theta s}\,ds$$

$$= \frac{\theta\lambda^k}{k!}\int_0^\infty s^k e^{-s(\lambda+\theta)}\,ds \qquad \left[\begin{array}{l} u = s(\lambda+\theta) \\ du = (\lambda+\theta)\,ds \end{array}\right]$$

$$= \frac{1}{(\lambda+\theta)^k}\cdot\frac{\theta\lambda^k}{k!}\left(\frac{1}{(\lambda+\theta)}\right)\int_0^\infty u^k e^{\overbrace{-s(\lambda+\theta)}^{-u}}\,du$$

$$= \frac{\theta\lambda^k}{k!(\lambda+\theta)^{k+1}}\Gamma(k+1) = \theta\left(\frac{\lambda^k}{(\lambda+\theta)^{k+1}}\right).$$

The following will be useful for the 5.4 work. Suppose $X(t)$ is a Poisson process of rate λ. What is the conditional PDF for the first arrival W_1 assuming $X(t)=n$? For $n=1$, Theorem 5.7 tells us that the conditional density is $f(w_1) = \frac{1}{t}$ for $w_1 \in [0,t]$ (the uniform density). For general n, the conditional PDF is

$$f(w_1) = \int_{w_1}^{t}\int_{w_2}^{t}\cdots\int_{w_{n-1}}^{t} dw_n \cdots dw_3\, dw_2 \text{, or}$$

$$f(w_1) = n/t^n \,(t-w_1)^{n-1} \quad \text{for } w_1 \in [0,t].$$

The first equality uses Theorem 5.7. The second may be checked inductively.

E 5.4.1 $E[W_1 \mid X(1)=n]$

$$= \int_0^1 w_1 \cdot [\,n(1-w_1)^{n-1}\,]\,dw_1 \qquad \begin{bmatrix} s = w_1 \\ z = 1-s \\ dz = -ds \end{bmatrix}$$

$$= -n\int_1^0 (1-z)\,z^{n-1}\,dz$$

$$= -n\left[\frac{z^n}{n} - \frac{z^{n+1}}{n+1}\right]\Big|_1^0$$

$$= n\left[\frac{1}{n} - \frac{1}{n+1}\right] = \frac{1}{n+1}.$$

E 5.4.2 Using Theorem 5.7,

$E[W_1W_2]$ (given $X(1)=2$)

$$= \int_0^1\int_{w_1}^1 2w_1w_2\,dw_2\,dw_1$$

$$= \int_0^1 w_1 w_2^2\Big|_{w_1}^1 dw_1$$

$$= \int_0^1 w_1 - w_1^3\,dw_1$$

$$= \frac{w_1^2}{2} - \frac{w_1^4}{4}\Big|_0^1 = \frac{1}{4}.$$

E 5.4.5 For a single customer, the answer would be

$$\int_0^1\int_0^{1-w_1} \alpha e^{-\alpha s}\,ds\,dw_1$$

$$= \int_0^1 -e^{-\alpha s}\Big|_0^{1-w_1} dw_1$$

$$= \int_0^1 \left[1 - e^{-\alpha(1-w_1)}\right]dw_1$$

$$= 1+\left[-e^{-\alpha}\left[\frac{e^{\alpha w_1}}{\alpha}\right]\Big|_0^1\right]$$

$$= 1 - \frac{1}{\alpha} + \frac{e^{-\alpha}}{\alpha}$$

$$= 1 - \frac{1-e^{-\alpha}}{\alpha}.$$

For five customers, the answer is $\left[1 - \frac{1-e^{-\alpha}}{\alpha}\right]^5$.

P5.4.5 Write $n = \beta t$. We have

$$\lim_{t\to\infty} f(w_1) = \lim_{t\to\infty} \frac{\beta t}{t^n}(t-w_1)^{\beta t-1}$$
$$= \lim_{t\to\infty} \beta t/_{t^{\beta t}}(t-w_1)^{\beta t-1}$$
$$= \beta \lim_{t\to\infty} \left[1 - w_1/t\right]^{\beta t-1}$$
$$= \beta \lim_{t\to\infty} \left[\left[1-w_1/t\right]^t\right]^\beta \left[1-w_1/t\right]^{-1}$$
$$= \beta e^{-\beta w_1}.$$

Thus the limiting dist. of W_1 is exponential!

P5.4.6

(a)
$$E[W_1 \mid X(t)=2] = \int_0^t w_1\left[2/t^2(t-w_1)\right]dw_1$$
$$= 2/t^2 \cdot \left[t w_1^2/2 - w_1^3/3\right]\Big|_0^t$$
$$= (2/t^2)(t^3/2 - t^3/3)$$
$$= t/3.$$

(b)
$$E[W_3 \mid X(t)=5] = \frac{5!}{t^5}\int_0^t\int_{w_1}^t\int_{w_2}^t\int_{w_3}^t\int_{w_4}^t w_3\, dw_5\, dw_4\, dw_3\, dw_2\, dw_1$$
$$= \frac{5!}{t^5}\int_0^t\int_{w_1}^t\int_{w_2}^t\int_{w_3}^t w_3(t-w_4)\, dw_4\, dw_3\, dw_2\, dw_1$$
$$= 5!/t^5\left[\int_0^t\int_{w_1}^t\int_{w_2}^t \left[w_3 t w_4 - w_3 w_4^2/2\right]\Big|_{w_3}^t dw_3\, dw_2\, dw_1\right.$$
$$= 5!/t^5\int_0^t\int_{w_1}^t\int_{w_2}^t (w_3 t^2 - w_3 t^2/2) - (w_3^2 t - w_3^3/2)\, dw_3\, dw_2\, dw_1$$
$$= 5!/t^5\int_0^t\int_{w_1}^t \left[\tfrac{1}{2}t^2 w_3^2/2 - t w_3^3/3 + w_3^4/8\right]\Big|_{w_2}^t dw_2\, dw_1$$
$$= 5!/t^5\int_0^t\int_{w_1}^t \left[t^4/4 - t^4/3 + t^4/8 - \left(\frac{t^2 w_2^2}{4} - \frac{t w_2^3}{3} + \frac{w_2^4}{8}\right)\right]dw_2\, dw_1$$
$$= 5!/t^5\int_0^t\left[\int_{w_1}^t\left[t^4/24 - \frac{t^2 w_2^2}{4} + \frac{t w_2^3}{3} - w_2^4/8\right]dw_2\right]dw_1$$
$$= 5!/t^5\int_0^t\left[t^4/24\, w_2 - \frac{t^2 w_2^3}{12} + \frac{t w_2^4}{3\cdot 4} - w_2^5/40\right]\Big|_{w_1}^t dw_1$$
$$= 5!/t^5\int_0^t\left[t^5/60 - \left(\frac{t^4 w_1}{24} - \frac{t^2 w_1^3}{12} + \frac{t w_1^4}{12} - w_1^5/40\right)\right]dw_1$$
$$= 5!/t^5\left[t^5/60\, w_1 - t^4/24\,(w_1^2/2) + t^2/12\,(w_1^4/4) - t w_1^5/60 + w_1^6/240\right]\Big|_0^t$$
$$= (5!/t^5)(t^6/240) = \boxed{t/2}.$$